An interesting genus 1 pencil

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June 16, 2009

1 Introduction

The purpose of this note is to construct a pencil X over $\mathbb{A}^1_{\mathbb{Q}}$ with the following properties:

- The generic fiber is a smooth curve of genus 1.
- $X(\mathbb{Q}) = \varnothing$.
- $X(\mathbb{Q}_v) \neq \varnothing$ for all places v of $\mathbb{Q}$.
- No rational fiber of X is locally trivial.

The idea is to start with an isotrivial family of elliptic curves—specifically, that given by $E : y^2 = x^3 - 4x$—and explicitly construct a cocycle

$$\xi : \mathrm{Gal}(\mathbb{Q}(t, \sqrt{d(t)})/\mathbb{Q}(t)) \to \langle T \rangle$$

for some 2-torsion point T and some $d \in \mathbb{Q}(t)^*$. Each fiber gives a torsor for $y^2 = x^3 - 4x$ over $\mathbb{Q}$. We show that

1. for each v, there is some t such that $X_t(\mathbb{Q}_v) \neq \varnothing$;
2. for each t, there is some v such that $X_t(\mathbb{Q}_v) = \varnothing$.

2 Construction of the pencil

As in the introduction, let E be given by $y^2 = x^3 - 4x$, and let T be the 2-torsion point $(2, 0)$. Let $d(t) = t^3 + 5t - 10$. Let $K = \mathbb{Q}(t)$ and $L = K(\sqrt{d})$. Write σ for the nontrivial automorphism of L over K. Pull back E to $\mathbb{A}^1_{\mathbb{Q}}$, where we consider K to be the function field of $\mathbb{A}^1_{\mathbb{Q}}$. Finally, let X be the $\mathbb{A}^1_{\mathbb{Q}}$-torsor obtained via the cocycle

$$\xi(\sigma) = T.$$

(That is, ξ has splitting field L.) It has equation

$$dw^2 = d^2 - 12dz^2 + 4z^4$$

(see [2, p.301]).

Proposition 2.1. *Let S be a finite set of primes. Then there exists $t \in \mathbb{Q}$ such that for all $v \in S$, $X_t(\mathbb{Q}_v) \neq \varnothing$.*

Proof. Let $p_1, \ldots, p_n$ be the prime numbers corresponding to the finite places in S; if the archimedean prime is in S, choose the p_i to be positive. Let $t = \Pi p_i^{-2}$. Then $d(t)$ is a square in $\mathbb{Q}_v$ for all v in S. In particular, X_t over $\mathbb{Q}_v$ is isomorphic to $E_{\mathbb{Q}}$. □

Lemma 2.2. *For every $t \in \mathbb{Q}$, there exists a prime v such that $X_t(\mathbb{Q}_v) = \varnothing$.*

Proof. The elliptic curve $y^2 = d(t)$ has trivial Mordell-Weil group over $\mathbb{Q}$, so for each $t \in \mathbb{Q}$, there exists a place v for which $d(t)$ is a nonsquare in $\mathbb{Q}_v$. Fixing t, we observe that X_t is the torsor for $E_{\mathbb{Q}}$ given by $\xi(\sigma) = (2, 0)$, where σ is the nontrivial automorphism of $\mathbb{Q}(\sqrt{d(t)})$ over $\mathbb{Q}$. If $v \neq 2, \infty$, then since the cocycle ξ is ramified at v and E has conductor 64, we have $X_t(\mathbb{Q}_v) = \varnothing$.

If $v = \infty$, the equation for X_t over $\mathbb{R}$ is isomorphic to

$$-w^2 = 1 + az^2 + bz^4$$

for some $a, b > 0$; clearly $X_t(\mathbb{R}) = \varnothing$.

Now suppose $d(t) \equiv 2 \pmod{\mathbb{Q}^{*2}}$. Then $\mathbb{Q}(\sqrt{d(t)}) = \mathbb{Q}(\sqrt{2})$. The equation for X_t is isomorphic to

$$2w^2 = 1 - 6z^2 + z^4.$$

(We obtain this equation by first translating $(2, 0)$ to $(0, 0)$, so that the equation for E becomes $y^2 = x^3 + 6x + 8$. Then we use the formula from [2, p.301].) Consider this equation over $\mathbb{Q}_2$. Let v be the valuation on $\mathbb{Q}_2$, normalized so that $v(2) = 1$. Clearly, the left hand side must have odd valuation. If $v(z) \neq 0$, then the right hand side has even valuation, and there is no solution. Suppose that $v(z) = 0$. We see that $z^2 \equiv 1 \pmod 8$, so that the right hand side is congruent to

$$1 - 6 + 1 \equiv 4 \pmod 8;$$

again, the right hand side has even valuation, and there is no solution. □

Proposition 2.3. *X is locally trivial, but no fiber of X is locally trivial. In particular, $X(\mathbb{Q}) = \varnothing$.*

References

[1] John Cremona, *Elliptic curve data.*

[2] Joseph H. Silverman, *The arithmetic of elliptic curves*, Springer-Verlag, New York, 1992, Corrected reprint of the 1986 original. MR 95m:11054