Extensions to WC IV

Pete L. Clark Shahed Sharif

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1 Disclaimer

Understanding of the following depends on an acquaintance with Clark's [Claar], available at `http://math.uga.edu/~pete/trans.pdf`.

2 The Laurent series ring $\mathbb{C}((t))$

Let $\mathbb{C}_g$ be the Laurent series ring in g variables. In this section, we will prove the following two results.

Theorem 2.1. *$\mathbb{C}_1$ is not $\mathbf{WC}(i)$ for any i.*

Theorem 2.2. *$\mathbb{C}_1$ is almost $\mathbf{WC}(1)$.*

As observed in Clark's WCIV, Theorem 2.2 follows from the following:

Theorem 2.3. *Let $A/\mathbb{C}_1$ be an abelian variety with semi-abelian reduction. Let X be a principal homogeneous space for A with period P. If P is odd, then the index of X is P. If P is even, then the index of X divides $2P$.*

Proof of Theorem 2.2. Fix d, the dimension of the abelian varieties. There is a constant $f = f(d)$ such that every abelian variety of dimension d has semi-abelian reduction over an extension $L/\mathbb{C}_1$ of degree less than or equal to d. It immediately follows from Thm. 2.3 that the index is less than or equal to $2f \cdot P$ for every principal homogeneous space for A over $\mathbb{C}_1$. □

2.1 Shafarevich duality on strictly local 1-dimensional rings

The key ingredient in the proofs of both theorems is a duality constructed by Shafarevich for abelian varieties over fields K which are the fraction field of a complete dvr with algebraically closed residue field k; in our case, $k = \mathbb{C}$. In fact, both theorems hold true in this more general context away from the characteristic p of k; that is, there is a constant $f(d)$ depending only on the dimension of an abelian variety A such that every principal homogeneous space over A with period P and index I, for which P is not divisible by p, one has $I \leq f(d)P$.

For the remainder of this section, we assume that K is as above. Let G denote the absolute Galois group of K. We first prove a structure theorem for abelian varieties over K.

Theorem 2.4. *Let A be an abelian variety over K. Then $A(K)$ is an extension of a finite group Φ_K by a group $A_0(K)$, where $A_0(K)$ is divisible by every n coprime to char k.*

Proof. This is just an application of Chevalley's theorem plus the Néron mapping property. That is, let $\mathscr{A}$ be the Néron model for A, and $A_k := \mathscr{A}_k$ the special fiber of $\mathscr{A}$. The special fiber $\mathscr{A}_k$ is a quasi-projective commutative algebraic group over an algebraically closed field, and therefore is an extension of a finite group by an n-divisible group; indeed, the latter group is the connected component of the identity. Let $A_0(K)$ be the subgroup of points whose reduction lies in the identity component of A_k.

To show $A_0(K)$ is n-divisible, we apply the Néron mapping property: given $x \in A(K)$ which reduces to $\overline{x}$ in the identity component of $\mathscr{A}_k$, find $\overline{y} \in \mathscr{A}_k$ such that $n\overline{y} = \overline{x}$. The Néron mapping property gives a point $y \in A(K)$ such that $ny - x$ lies in the kernel of reduction. But the kernel of reduction is a formal group, and therefore is n-divisible. □

Let $\hat{A}$ be the dual abelian variety to A. We now define our pairing

$$\varepsilon : H^1(K, A[n]) \times \hat{A}(K)[n] \to \mathbb{Z}/n\mathbb{Z}$$

as follows. The cup product gives a class in $H^1(K, A[n] \otimes \hat{A}[n])$. Composing with the Weil pairing yields an element of $H^1(K, \mu_n) = \mathrm{Hom}(G, \mu_n)$. Fix a uniformizer t for the discrete valuation ring of K, and choose any nth root $t^{1/n}$. There is a canonical character χ given by

$$\chi(\tau) = \frac{\tau t^{1/n}}{t^{1/n}}.$$

Observe that χ is independent of the choice of t. Thus, every element of $\mathrm{Hom}(G, \mu_n)$ is a multiple of χ, given by $a\chi$ for some $a \in \mathbb{Z}/n\mathbb{Z}$. From this, $\mathrm{Hom}(G, \mu_n) = \mathbb{Z}/n\mathbb{Z}$. Taken together with the cup product and the Weil pairing, we have defined the bilinear pairing ε.

Proposition 2.5. *The pairing ε induces a pairing*

$$\varepsilon_0 : H^1(K, A)[n] \times \hat{A}_0(K)[n] \to \mathbb{Z}/n\mathbb{Z}.$$

Proof. Recall the Kummer sequence for A

$$0 \to \frac{A(K)}{nA(K)} \to H^1(K, A[n]) \to H^1(K, A)[n] \to 0.$$

From this, it suffices to show that ε restricted to $\frac{A(K)}{nA(K)} \times \hat{A}_0(K)[n]$ is trivial.

Write δ for the first map in the exact sequence above. Given $a \in A(K)$, we will abbreviate $\delta(a \pmod{nA(K)})$ to simply $\delta(a)$. Then $\delta(a)(\tau)$ is given by $\tau b - b$ for any $b \in A(\overline{K})$ for which $nb = a$. By Thm. 2.4, there is an integer m such that $ma \in A_0(K)$. This implies that $m\delta(a) = 0$; in particular, we may assume that $p \nmid m$.

Since $A_0(K)$ is divisible, there is $b' \in A_0(K)$ such that $mnb' = ma$. Replacing b with $b - b'$ does not change $\delta(a)$; thus we may assume that $b \in A[mn]$.

Now let $x \in \hat{A}_0(K)[n]$. By the divisibility of $\hat{A}_0(K)$, there is $y \in \hat{A}_0(K)[mn]$ such that $my = x$.

It remains to show that $\varepsilon(\delta(a), x) = 0$. It suffices to show $e_n(\tau b - b, x) = 1$ for all $\tau \in G$, where e_n is the level n Weil pairing. We have

$$\begin{aligned} e_n(\tau b - b, x) &= e_n(\tau b - b, my) \\ &= e_{mn}(\tau b - b, y) \\ &= \frac{e_{mn}(\tau b, y)}{e_{mn}(b, y)} \\ &= 1 \end{aligned}$$

where the last equality follows from the Galois-equivariance of the Weil pairing. □

Proposition 2.6. *The right kernel of ε_0 is trivial.*

Proof. In other words, we wish to show that if

$$\varepsilon_0(\xi, x) = 0$$

for all $\xi \in H^1(K, A)[n]$, then $x = 0$. Since the pairing takes values in $\mathrm{Hom}(G, \mu_n)$, it suffices to show that $e_n(\eta(\tau), x) = 1$ for every $\eta \in Z^1(K, A[n])$, $\tau \in G$, where Z^1 is the set of 1-cocycles. The Weil pairing is nondegenerate, so in fact we need only show that the set of all $\eta(\tau)$ is in fact all of $A[n]$. Under our hypothesis that $p \nmid n$, $A[n]$ is tamely ramified, so the action of G on $A[n]$ factors through its tame quotient G^t. The group G^t is pro-cyclic; choose τ to be a topological generator. Choose L/K a tamely ramified extension of large degree—say, the degree n extension over $K(A[n])$. The cohomology theory of cyclic groups tells us that

$$Z^1(L/K, A[n]) = \ker \mathrm{Nm}$$

where $\mathrm{Nm} : A(L)[n] \to A(K)[n]$ is given by the operator $1 + \tau + \cdots + \tau^{[L:K]-1}$, and the isomorphism is given by $\eta \mapsto \eta(\tau)$. By our choice of L, $\ker \mathrm{Nm} = A[n]$. The inflation map gives $Z^1(L/K, A[n])$ as a subgroup of $Z^1(K, A[n])$, from which the proposition follows. □

Theorem 2.7. *ε_0 is a perfect pairing.*

Proof. In light of Proposition 2.6, it suffices to show that

$$\#\hat{A}_0(K)[n] = \#H^1(K, A)[n].$$

The exact sequence

$$0 \to A_0(K) \to A(K) \to \Phi_K \to 0$$

gives rise, via the snake lemma and the n-divisibility of $A_0(K)$, to the exact sequences

$$0 \to A_0(K)[n] \to A(K)[n] \to \Phi_K[n] \to 0$$
$$0 \to \frac{A(K)}{nA(K)} \to \frac{\Phi_K}{n\Phi_K} \to 0$$

The first sequences tells us that

$$\#A_0(K)[n] = \frac{\#A(K)[n]}{\#\Phi_K[n]}.$$

Observe that since Φ_K is a finite group, $\Phi_K[n]$ and $\Phi_K/n\Phi_K$ have the same order. The second sequence plus the latter observation shows that $\frac{A(K)}{nA(K)}$ has finite order, equal to $\#\Phi_K[n]$. Therefore

$$\#A_0(K)[n] = \frac{\#A(K)[n]}{\#\frac{A(K)}{nA(K)}}.$$

From our description in the proof of Proposition 2.6 of $Z^1(K, A[n])$, we have

$$H^1(K, A[n]) = \frac{A[n]}{(1-\tau)A[n]}.$$

But from

$$0 \to A(K)[n] \to A[n] \to (1-\tau)A[n] \to 0,$$

where the second map is $(1-\tau)$, we see that

$$\#H^1(K, A[n]) = \#A(K)[n].$$

Now consider the Kummer sequence

$$0 \to \frac{A(K)}{nA(K)} \to H^1(K, A[n]) \to H^1(K, A)[n] \to 0.$$

From this,

$$\#H^1(K, A)[n] = \frac{\#H^1(K, A[n])}{\#\frac{A(K)}{nA(K)}}.$$

Combining with the above equality, we obtain

$$\#H^1(K, A)[n] = \#A_0(K)[n].$$

Lastly, A_0 and $\hat{A}_0$ are the identity components of isogenous commutative group schemes, hence

$$\#A_0(K)[n] = \hat{A}_0(K)[n]$$

and the theorem follows. □

Proposition 2.8. *Let L/K be a finite Galois extension. Under the pairing ε_0, $\mathrm{res}_{L/K}$ and $\mathrm{Nm}_{L/K}$ are adjoint; that is, the following diagram commutes:*

$$\begin{array}{ccc}
H^1(K,A)[n] \times \hat{A}_0(K)[n] & \xrightarrow{\varepsilon_0} & \frac{\mathbb{Z}}{n\mathbb{Z}} \\
\downarrow \text{res} \quad \uparrow \text{Nm} & & \downarrow = \\
H^1(L,A)[n] \times \hat{A}_0(L)[n] & \xrightarrow{\varepsilon_0} & \frac{\mathbb{Z}}{n\mathbb{Z}}.
\end{array}$$

Proof. It suffices to consider the case where L/K is tamely ramified. In this case L/K is cyclic. Let $[L : K] = r$ and let τ be a topological generator for the tame inertia group. For $\xi \in H^1(K, A[n])$, let $f \in Z^1(K, A[n])$ represent it. Then $\operatorname{res}\xi$ is represented by $\operatorname{res} f$. We have $\varepsilon_0(\xi, \operatorname{Nm} x)$ is determined by

$$e_n(f(\tau), \operatorname{Nm} x) = \sum_{i=0}^{r-1} e_n(f(\tau), \tau^i x).$$

On the other hand, $\varepsilon_0(\operatorname{res}\xi, x)$ is determined by computing $e_n(f(\tau^r), x)$. Since f is a 1-cocycle,

$$f(\tau^r) = \sum_{i=0}^{r-1} \tau^i f(\tau).$$

The result now follows by the Galois-equivariance and bilinearity of the Weil pairing. □

3 Weil extension of scalars and the proof of Theorem 2.1

In [Sha61, p.97], Shafarevich showed that $\mathbb{C}_1$ is not $\mathbf{WC}(1)$. The proof is adapted easily to show that $\mathbb{C}_1$ is not $\mathbf{WC}(i)$ for any i.

Proof of Theorem 2.1. Observe that the Galois group of $K = \mathbb{C}((t))$ is procyclic; the unique degree m extension, for m a positive integer, is $K(t^{1/m})$. Let P be a positive integer. The Shafarevich duality between $H^1(K, A)[n]$ and $\hat{A}_0(K)[n]$, along with Prop. 2.8, allows us to reduce the theorem to the following statement: given i, there exists an abelian variety A such that the norm map

$$\mathrm{Nm}_{L/K} : A(L)[P] \to A(K)[P]$$

is nonzero, where $L = K(t^{1/P^i})$. (This A is really the $\hat{A}$ in the notation of our Shafarevich duality.)

We construct A as follows. Let E/K be an elliptic curve with at least one rational point of exact order n; the easiest examples are isotrivial elliptic curves. Let A be the Weil restriction of scalars for E from L to K. One way to construct

A is via Galois descent on E^{P^i} equipped with the following twisted Galois action: if τ is a topological generator for G, then τ acts on $(x_1, \ldots, x_{P^i}) \in E^{P^i}$ via

$$\tau(x_1, \ldots, x_{P^i}) = (\tau x_2, \ldots, \tau x_{P^i}, \tau x_1).$$

That is, the twisting consists of cyclically permuting the factors. In particular, if x_1 is the nontrivial n-torsion point on E, and $x_i = 0$ for $i \geq 2$, then

$$\mathrm{Nm}(x_1, 0, \ldots, 0) = (x_1, x_1, \ldots, x_1).$$

The theorem follows. □

4 Monodromy on semi-abelian varieties

In this section, we will assume that K is the fraction field of a strictly henselian dvr R, and that A/K is an abelian variety such that the Néron model for A has semi-abelian reduction. Finally, let ℓ be a prime not equal to the residue characteristic p of K; for $K = \mathbb{C}((t))$, this is no condition at all.

Our starting point will be the following well-known theorem on the ℓ-adic monodromy representation on A.

Theorem 4.1. *Let A/K be as above. Let $T_\ell(A(\overline{K}))$ be the ℓ-adic Tate module attached to A. Then the action of the absolute Galois group G of K on $T_\ell(A(\overline{K}))$ factors through its maximal pro-ℓ quotient. If τ is a topological generator for G, and τ acts on $T_\ell(A(\overline{K}))$ via $\alpha \in \mathrm{Aut}(T_\ell(A(\overline{K})))$, then*

$$(\alpha - 1)^2 = 0.$$

Proof. See [sga72, Exp. IX, Cor. 3.5.2]. □

Lemma 4.2. *Let n be a positive integer.*

1. *In $\mathbb{Z}[x]$, $(x-1)^2 \mid (x^n - nx + n - 1)$.*
2. $\displaystyle\sum_{m=1}^{n} x^m \equiv \frac{n(n+1)}{2}(x-1) + n \pmod{(x-1)^2}$.

Proof. For the first claim, letting $p(x) = x^n - nx + n - 1$, one verifies that $p(1) = p'(1) = 0$.

For the second claim, we use the first part of the lemma to obtain

$$x^m \equiv mx - m + 1 \pmod{(x-1)^2}.$$

Summing the right-hand side over all m, we obtain the formula stated above. □

Proposition 4.3. *Let A/K be as above. Let n be a positive integer not divisible by the residue characteristic of K. If n is odd, let L be the unique degree n extension over K. If n is even and the residue characteristic is not 2, let L be the unique extension of degree $2n$. Then the norm*

$$\mathrm{Nm} : A(L)[n] \to A(K)[n]$$

is the zero map.

Proof. The action of Nm on $A[n]$ is now given by the second part of Lemma 4.2; namely, in the case that n is odd,

$$\mathrm{Nm} = \sum_{m=1}^{n} \alpha^m = \frac{n(n+1)}{2}(\alpha - 1) + n.$$

But the latter expression is 0 mod n. When n is even, one instead has

$$\mathrm{Nm} = \sum_{m=1}^{2n} \alpha^m = \frac{2n(2n+1)}{2}(\alpha - 1) + 2n$$

which again is congruent to 0. □

Proof of Thm. 2.3. The theorem follows immediately from Shafarevich duality and the above proposition. □

References

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