HW Selected Solutions

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5.3 Even though we hadn't done the Residue Theorem yet, now that we have it, I will use it where appropriate.

For (f), we rewrite this as

$$\frac{i^z}{i^3} = -i \cdot i^z = -i \cdot e^{z \log i} = -ie^{iz \cdot \pi/2}.$$

This is an entire function, and hence the integral is zero.

For (h), the function is holomorphic except at -4 and $\pm i$. Since -4 is outside the circle, it does not impact the integral. Note that $\pm i$ are simple zeroes of the denominator. Writing our function as

$$\frac{\frac{1}{(z+4)(z+i)}}{z-i},$$

we apply Prop 9.14 to see that the residue at i is $\frac{1}{(i+4)(2i)} = -\frac{1}{34}(1+4i)$. By similar reasoning, the residue at $-i$ is $-\frac{1}{34}(1-4i)$. The sum of the residues is $-\frac{1}{17}$, and so the value of the integral is $-\frac{2\pi i}{17}$.

For (i), the integrand has a simple pole at 2 and a double pole at 1. Via Prop 9.14, the residue at 2 is

$$\exp(4)/(2-1)^2 = e^4.$$

For the residue at 1, I will use Prop 9.11b with $n = 2$. We first compute the derivative of $\exp(z)/(z-2)$:

$$\begin{aligned} \frac{d}{dz}\frac{\exp(z)}{(z-2)} &= \frac{\exp(z)(z-2) - \exp(z)}{(z-2)^2} \\ &= \frac{z\exp(z) - 3\exp(z)}{(z-2)^2}. \end{aligned}$$

Then we plug in $z = 1$ to get

$$\frac{\exp(1) - 3\exp(1)}{(1-2)^2} = -2e.$$

The sum of the residues is $e^4 - 2e$. The Residue Theorem now tells us the value of the integral is $2\pi i(e^4 - 2e)$.

5.10 As discussed in the hint, we let $L = \lim_{z\to\infty} f(z)$, set $\varepsilon = 1$, and so have $R > 0$ such that $|z| > R$ implies $|f(z) - L| < 1$. This implies that for $|z| > R$, $|f(z)| < |L| + 1$. Since f is continuous, so is $|f|$. Now $\overline{D}[0, R]$ is a compact (closed and bounded), and on such sets, continuous functions attain their maximum. In particular, $|f|$ attains its maximum, call it K, on $\overline{D}[0, R]$. Let $M = \max(K, |L| + 1)$. It follows that for all z, $|f(z)| \leq M$, proving the claim.

7.20 For (a), let $\varepsilon > 0$. By the uniformity, $\exists N$ such that $n \geq N$ implies $|f_n(z)| < \varepsilon$ for all $z \in G$; in particular, this holds for $z = z_n$. In other words, $|f_n(z_n)| < \varepsilon$ for $n \geq N$. The claim follows.

For (b), we have $f_n(e^{-1/n}) = e^{-1}$ which does not go to zero as $n \to \infty$. By (a), the sequence f_n does not converge uniformly to 0.

7.22 Both (a) and (b) converges on $D[0, 1]$, by Prop 7.7a; for (a), $c = |1/z|$ when $z \neq 0$ and $p(n) = n$, while for (b), c is the same but $p(n) = \frac{1}{n}$.

(a) diverges for $|z| \geq 1$, since then $|nz^n| \geq n \to \infty$. Thus the sequence converges to 0 on $D[0, 1]$. The convergence is not uniform; using 7.20 with $z_n = e^{-1/n}$, we get $f_n(z_n) = ne^{-1} \to \infty$. The conclusion follows.

(b) diverges for $|z| > 1$, but for $|z| = 1$, the sequence converges to 0: $|z|^n/n = \frac{1}{n} \to 0$. The convergence on $\overline{D}[0, 1]$ is uniform. Given $\varepsilon > 0$, choose N so that $1/N < \varepsilon$. Then for $n \geq N$, on the closed disk we have

$$\begin{aligned} |z^n/n| &= |z|^n/n \\ &\leq 1^n/N \\ &= 1/N \\ &< \varepsilon. \end{aligned}$$

(c) converges on the entire region. Certainly we have convergence at 0. If $z \neq 0$, choose N so that $N|z| > 1$. Then for $n \geq N$,

$$|1 + nz| \geq n|z| - 1 \to \infty$$

since $|z| > 0$. This means that $|\frac{1}{1+nz}| \to 0$ in the region, except for $z = 0$. Notice that the limit at $z = 0$ is 1, so the limit function is discontinuous at $z = 0$. Therefore the convergence is not uniform.