HW 9 Selected Solutions

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6.1 The Laplace equation is not difficult to verify. The tricky part is showing that you can take arbitrarily many partials, and that these partials are continuous. For this, we have to use the fact that holomorphic functions have arbitrarily many derivatives.

To that end, let u be harmonic at a point z_0, so that it satisfies the Laplace equation and has continuous 2nd order partials in a disk D around z_0. Since disks are simply connected, u is the real part of a holomorphic function f on D; say $f(z) = u(x,y) + iv(x,y)$. By Corollary 5.5, all partials of f exist and are continuous, and hence the same holds of u.

Finally, since the partials are continuous, they commute, and in particular they commute with the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

In other words, if $\mathscr{P}$ is a partial differentiation operator,

$$\Delta(\mathscr{P}(u)) = \mathscr{P}(\Delta u) = \mathscr{P}(0) = 0.$$

Therefore $\mathscr{P}(u)$ is harmonic.

6.3 There are of course many examples, but x^2 is a relatively simple one: $p(x) = x$ is harmonic since $p_{xx} = p_{yy} = 0$, but $p \cdot p = x^2$ is not harmonic since $\Delta x^2 = 2 + 0 = 2 \neq 0$.

6.6 This can be done directly using the Laplace equation, but it's ugly. Instead, let $z = x + iy$ and let $p(x,y) = \ln|f(x,y)|$. Let $z_0 \in G$ and $w_0 = f(z_0)$. As $w_0 \neq 0$, $\exists \varepsilon > 0$ such that $0 \notin D[w_0, \varepsilon]$. But f is continuous, so $\exists \delta$ such that if $z \in D[z_0, \delta]$, then $f(z) \in D[w_0, \varepsilon]$. This is significant because there is a branch of the logarithm $\mathcal{L}\text{og}$ holomorphic on $D[w_0, \varepsilon]$, so on $D[z_0, \delta]$, $\mathcal{L}\text{og}(f)$ is holomorphic. But the real part of $\mathcal{L}\text{og}(f)$ is $\ln|f|$ (only the imaginary part changes if we change the branch). Therefore at z_0, $\ln|f|$ is harmonic. But this holds for all $z_0 \in G$, and so the claim follows.

(The subtleties with the ε and δ are required because if we stray too far from z_0, then we may have to change the branch. For example, if $f(z) = z$ and $G = \mathbb{C} - \{0\}$, then as long as z is not on the negative real axis, we can use the principal branch. But if z *is* on the negative real axis, we have to switch to a different branch.)