HW 8 Selected Solutions

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5.1ab For (a), this is $\pi i f''(0)$ where $f(z) = \exp(z^2)$, and since $f''(z) = 2\exp(z^2) + 4z^2\exp(z^2)$, we get $\pi i f''(0) = 2\pi i$.

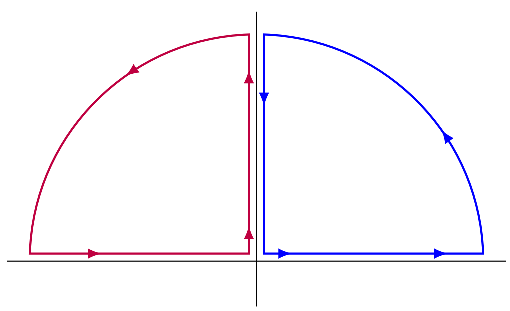
For (b) this is $2\pi i g'(\pi i)$ where $g(z) = \exp(3z)$, and thus the answer is $6\pi i \exp(3\pi i) = -6\pi i$.

5.13 Let $w \in \mathbb{C}$, $r > 0$, and $\gamma = C[w, r]$. Then by the Cauchy Integral Formula,

$$\begin{aligned}|f'(w)| &= \frac{1}{2\pi}\left|\int_\gamma \frac{f(z)}{(z-w)^2}\,dz\right| \\ &\leq \frac{1}{2\pi}\max_{z\in\gamma}\frac{|f(z)|}{|z-w|^2}\operatorname{len}(\gamma) \\ &\leq r\max_{z\in\gamma}\frac{\sqrt{|z|}}{r^2} \\ &= \frac{1}{r}\sqrt{|w|+r}.\end{aligned}$$

Now we let $r \to \infty$. The last expression above goes to 0, and hence $|f'(w)| = 0$, so $f' = 0$, and hence f is a constant. But $|f(0)| \leq \sqrt{0} = 0$, and so $f(0) = 0$. Therefore f is identically zero.

5.18 Let $R > 2$. Let γ_1 be the quarter-circular contour of radius R shown in blue below, and γ_2 to similar contour shown in purple. Let $\gamma = \gamma_1 + \gamma_2$; this will be a half-circular path. Finally, let σ be the curved part of γ; that is, the image of Re^{it} for $0 \leq t \leq \pi$.



First, we have that

$$\int_\gamma \frac{dz}{z^4+1} = \int_{-R}^{R}\frac{dx}{x^4+1} + \int_\sigma \frac{dz}{z^4+1}.$$

For the second integral,

$$\begin{aligned}\left|\int_\sigma \frac{dz}{z^4+1}\right| &\leq \max_{z\in\sigma} \frac{1}{|z^4+1|}\,\mathrm{len}(\sigma)\\ &\leq \frac{2\pi R}{R^4-1}.\end{aligned}$$

As $R \to \infty$, this goes to 0. Therefore

$$\lim_{R\to\infty} \int_\gamma \frac{dz}{z^4+1} = \int_{-\infty}^{\infty} \frac{dx}{x^4+1}.$$

It remains to evaluate the contour integral. We have

$$\int_\gamma \frac{dz}{z^4+1} = \int_{\gamma_1} \frac{dz}{z^4+1} + \int_{\gamma_2} \frac{dz}{z^4+1}.$$

Let $\omega = e^{i\pi/4}$. The roots of z^4+1 are $\omega, \omega^3, \omega^5$, and ω^7. Thus

$$\frac{1}{z^4+1} = \frac{\frac{1}{(z-\omega^3)(z-\omega^5)(z-\omega^7)}}{z-\omega} = \frac{\frac{1}{(z-\omega)(z-\omega^5)(z-\omega^7)}}{z-\omega^3}.$$

As ω is in the first quadrant and none of the others are, for γ_1 we can use the second expression above. As ω^3 is in the second quadrant and none of the others are, for γ_2 we can use the third expression above. Thus by the Cauchy Integral Formula,

$$\begin{aligned}\int_\gamma \frac{dz}{z^4+1} &= \int_{\gamma_1} \frac{dz}{z^4+1} + \int_{\gamma_2} \frac{dz}{z^4+1}\\ &= \frac{2\pi i}{(\omega-\omega^3)(\omega-\omega^5)(\omega-\omega^7)} + \frac{2\pi i}{(\omega^3-\omega)(\omega^3-\omega^5)(\omega^3-\omega^7)}.\end{aligned}$$

Some careful algebra shows that this equals $\frac{\pi}{\sqrt{2}}$.