HW 7 Selected Solutions

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4.23 For reflexivity, let $\gamma(t)$ be a closed path in G. Let $h(s,t) = \gamma(t)$. Since γ is continuous, so is h. Also, $\gamma \subset G$, so the image of h lies in G. Clearly $h(t,0) = h(t,1) = \gamma(t)$. Finally, $h(0,s) = \gamma(0) = \gamma(1) = h(1,s)$ for all s. Thus $\gamma \sim_G \gamma$.

For symmetry, let $\gamma_0 \sim_G \gamma_1$, so there exists a G-homotopy $h(s,t)$ between the two. Let $H(t,s) = h(t,1-s)$. The function H is continuous because h and $(t,s) \mapsto (t,1-s)$ is continuous, and the image of H is the same as that of h and hence lies in G. Finally, $H(t,0) = h(t,1) = \gamma_1(t)$, $H(t,1) = h(t,0) = \gamma_0(t)$, and $H(0,s) = h(0,1-s) = h(1,1-s) = H(1,s)$. Thus H gives a homotopy in the other direction; in other words, $\gamma_1 \sim_G \gamma_0$.

For transitivity, suppose $\gamma_0 \sim_G \gamma_1$ and $\gamma_1 \sim_G \gamma_2$. That means there exist homotopies h_0 between γ_0 and γ_1, and h_1 between γ_1 and γ_2. Define a new homotopy

$$h(t,s) = \begin{cases} h_0(t,2s) & 0 \le s \le \frac{1}{2} \\ h_1(t,2s-1) & \frac{1}{2} < s \le 1. \end{cases}$$

The image of both h_0 and h_1 lies in G, so the same is true of the image of h. Continuity is only unclear at $s = \frac{1}{2}$. But observe that

$$h\left(t, \frac{1}{2}\right) = h_0(t,1) = \gamma_1(t) = h_1(t,0),$$

and that $\lim_{(s,t)\to(1/2,t_0)} h(t,s) = \lim_{(s,t)\to(0,t_0)} h_1(t,s) = h_1(t_0,0)$ for all $t_0 \in [0,1]$. Thus h is continuous. Finally, we check the equalities necessary to be a homotopy:

$$\begin{aligned} H(t,0) &= h_0(t,0) = \gamma_0(t) \\ H(t,1) &= h_1(t,1) = \gamma_2(t) \\ H(0,s) &= h_i(0,2s-i) = h_i(1,2s-i) = H(1,s) \text{ for } i = 0,1. \end{aligned}$$

4.25 For (a), I will show that a closed path is homotopic to the constant path $\sigma(t) = 0$; that is, the origin. Given a closed path $\gamma(t)$, let $h(t,s) = s\gamma(t)$. Clearly this is continuous. We have $h(t,0) = 0$, while $h(t,1) = \gamma(t)$. Finally, $h(0,s) = s\gamma(0) = s\gamma(1) = h(1,s)$. It follows that $0 \sim_{\mathbb{C}} \gamma$, and by symmetry the result follows. (If you didn't want to use symmetry, you could also use $(1-s)\gamma(t)$; but I like my expression better!)

For (b), given γ_0, γ_1, we have by (a) that both are $\mathbb{C}$-homotopic to 0, but by transitivity, this means that they are $\mathbb{C}$-homotopic to each other.

4.29 Note that $z^6 - 1 = (z^3-1)(z^3+1)$, so the roots of z^3+1 are a subset of the roots of z^6-1. These latter roots are sixth roots of unity, which lie on the unit circle. Therefore if $r > 1$, then all roots of $z^3 + 1$ lie inside $C[0, r]$. In particular, letting G be $\mathbb{C}$ with the three roots of $z^3 + 1$ removed, if $r_0, r_1 > 1$, then $C[0, r_0] \sim_G C[0, r_1]$. (If you want a specific homotopy, do $\gamma_0(t) = r_0 e^{2\pi i t}$, $\gamma_1(t) = r_1 e^{2\pi i t}$, and

$$h(t, s) = ((1 - s)r_0 + s r_1)e^{2\pi i t}.$$

But I think this is overkill.)

As our integrand is holomorphic on G, Cauchy's Theorem shows that

$$\int_{C[0,2]} \frac{dz}{z^3 + 1} = \int_{C[0,r]} \frac{dz}{z^3 + 1}$$

for any $r > 1$.

Next, we obtain a bound. On $C[0, r]$, $|z^3 + 1| \geq |z|^3 - 1 = r^3 - 1$. Also, $\text{len}(C[0, r]) = 2\pi r$. Thus

$$\left| \int_{C[0,r]} \frac{dz}{z^3 + 1} \right| \leq \frac{1}{r^3 - 1} \cdot 2\pi r.$$

But as $r \to \infty$, this expression goes to 0. It follows that the original integral has value 0.