HW 6 Selected Solutions

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4.4 We parametrize $C[w, r]$ using $\gamma(t) = w + re^{it}$ with $0 \leq t \leq 2\pi$. Then

$$\begin{aligned}\int_\gamma \frac{dz}{z-w} &= \int_0^{2\pi} \frac{\gamma'(t)}{(w+re^{it})-w} dz \\ &= \int_0^{2\pi} \frac{ire^{it}}{re^{it}} dz \\ &= \int_0^{2\pi} i dz \\ &= it\Big|_0^{2\pi} = 2\pi i.\end{aligned}$$

Taking $w = 0$ and $r = 1$ yields the first case in the problem.

4.8 (a) This function has an antiderivative, $\frac{1}{3}z^3$. (To check, just differentiate; we know the power rule works for positive integer powers.) The endpoints of the path are $1 + i$ and 0, so the answer is

$$\frac{1}{3}((1+i)^3 - 0^3) = -\frac{2}{3} + \frac{2}{3}i.$$

(b) The function $\frac{1}{2}z^2$ is an antiderivative. Plugging in the endpoints yields

$$\frac{1}{2}((-1)^2 - (1)^2) = 0.$$

(c) The function $\exp(z)$ is an antiderivative. Evaluating, we get $\exp(z_0) - q1$.

(d) This function does *not* have an antiderivative! One way to check is to integrate over the top half of the unit disk; that is, the upper semicircle plus the diameter. If you do this, the integral is nonzero, so by Corollary 4.13, there is no antiderivative. But in any case, you didn't have to check this. Instead, one has to evaluate from the definition. Consider the path $\gamma(t) = 2(1-t) + t(3+i) = 2 + t + it$ for $t \in [0, 1]$. Then

$$\begin{aligned}\int_\gamma f &= \int_0^1 |2 + t + 3it|^2 (1+i)\,dt \\ &= (1+3i)\int_0^1 (2+t)^2 + t^2\,dt \\ &= (1+3i)\int_0^1 2t^2 + 4t + 4\,dt \\ &= (1+3i)\left[\frac{2}{3}t^3 + 2t^2 + 4t\right]_0^1 \\ &= \frac{20}{3}(1+3i).\end{aligned}$$

4.12 The trick here is to observe that $z^{\frac{1}{2}}$ has an antiderivative on $\mathbb{C} - \mathbb{R}^{\leq 0}$, precisely the region on which Log is holomorphic. So we will split the path into two pieces: a tiny piece straddling the negative x-axis, and the rest of the circle.

Choose $\varepsilon > 0$. Let σ be the piece of the circle given by $2e^{it}$ with $\pi - \varepsilon \leq t \leq \pi + \varepsilon$. Notice that -1 is contained in σ. Let ρ be the rest of the circle. It is parametrized by $2e^{it}$ with $-\pi + \varepsilon \leq t \leq \pi - \varepsilon$. The function $\frac{2}{3}z^{\frac{3}{2}}$ is an antiderivative for $z^{\frac{1}{2}}$ on ρ—one needs to verify this using the definition of complex exponentiation on p. 47 and the usual rules for differentiation, but I will omit it. Then

$$\int_\rho z^{\frac{1}{2}} = \left[\frac{2}{3}z^{\frac{3}{2}}\right]_{2e^{i(-\pi+\varepsilon)}}^{2e^{i(\pi-\varepsilon)}} = \frac{2}{3}(2e^{3\pi i/2 - 3\varepsilon i/2} - 2e^{-3\pi i/2 + 3\varepsilon i/2}).$$

As $\varepsilon \to 0$, this goes to $\frac{4}{3}(e^{3\pi i/2} - e^{-3\pi i/2}) = -\frac{8}{3}i$.

Now for the other piece! We have $\left|\int_\sigma z^{\frac{1}{2}}\right| \leq \max |z^{\frac{1}{2}}|\,\mathrm{len}(\sigma)$. Since we are on the circle of radius 2, $|z^{\frac{1}{2}}| = \sqrt{2}$ on σ. We have $\mathrm{len}(\sigma) = 4\varepsilon$. Therefore

$$\left|\int_\sigma z^{\frac{1}{2}}\right| \leq 4\sqrt{2}\varepsilon$$

which goes to 0 as $\varepsilon \to 0$.

The full integral is the sum of the two integrals, and so yields $-\frac{8}{3}i$.