HW 5 Selected Solutions

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3.32 If $\sin(z) = 0$, then $2i\sin(z) = 0$, so

$$\exp(iz) - \exp(-iz) = 0.$$

This means that $\exp(iz) = \exp(-iz)$. In particular, we must have $|\exp(iz)| = |\exp(-iz)|$. Write $z = x + iy$. Then

$$\begin{aligned} |\exp(-iz)| &= |\exp(-ix + y)| \\ &= |e^y| \\ &= e^y. \end{aligned}$$

Similarly, $|\exp(iz)| = e^{-y}$. Thus $e^y = e^{-y}$. Since e^t as a function $\mathbb{R} \to \mathbb{R}$ is injective, we must have $y = -y$, or $y = 0$. Therefore z is real.

As $\sin(z)$ agrees with the usual sine function when z is real, the rest of the problem follows.

3.41 (a) -1 (you can write $-1 + 0i$, but that is unnecessary)

(b) e^π

(c) We did this in class!

(d) $\sin i = \frac{1}{2i}(\exp(i^2) - \exp(-i^2)) = \frac{1}{2i}(e^{-1} - e) = \frac{i}{2}(e - e^{-1})$. This is a purely imaginary number, so when we exponentiate, we get

$$\cos\left(\frac{e - e^{-1}}{2}\right) + i\sin\left(\frac{e - e^{-1}}{2}\right).$$

(e) Log is a right inverse for exp, so we get $3 + 4i$

(f) This is $\exp(\frac{1}{2}\operatorname{Log}(1 + i))$. We have $|1 + i| = \sqrt{2}$ and $\operatorname{Arg}(1 + i) = \frac{\pi}{4}$, so

$$\frac{1}{2}\operatorname{Log}(1 + i) = \frac{1}{2}\ln(\sqrt{2}) + i\frac{\pi}{8} = \ln 2^{1/4} + i\frac{\pi}{8}.$$

Finally, we exponentiate to obtain

$$2^{1/4}\cos\frac{\pi}{8} + i2^{1/4}\sin\frac{\pi}{8}.$$

Explicit formulas for the cos and sin can be obtained using half-angle formulas, but that isn't necessary.

(g) $\sqrt{3} - i\sqrt{3}$ (You're not missing anything! The point is that $\sqrt{\ }$ has a specific meaning for real numbers.)

(h) We have $\frac{i+1}{\sqrt{2}} = e^{i\pi/4}$, so taking to the 4th yields $e^{i\pi} = -1$. (Note 3.51 which says for integer powers, we don't have to worry about branches of the logarithm.)

3.51 Suppose that b is an integer. Two branches of log differ by an integer multiple of $2\pi i$; say we have that $\log a$ could be the values z_0 and $z_0 + 2\pi i k$ with $k \in \mathbb{Z}$. Then

$$\begin{aligned}\exp(b(z_0 + 2\pi i k)) &= \exp(bz_0 + 2\pi i b k)\\ &= \exp(bz_0)\cdot\exp(2\pi i b k)\\ &= \exp(bz_0)\cdot(\exp(2\pi i))^{bk}\\ &= \exp(bz_0)\cdot(1)^{bk}\\ &= \exp(bz_0).\end{aligned}$$

Therefore in this case, the particular branch of log makes no difference to the value. Now suppose b is not an integer. Write $b = b_0 + \varepsilon$, where $b_0 = \lfloor b \rfloor$ and $\varepsilon = b - b_0$ is the fractional part; in particular, $0 < \varepsilon < 1$. We get

$$\begin{aligned}\exp(b(z_0 + 2\pi i k)) &= \exp(bz_0)\cdot\exp(2\pi i b k)\\ &= \exp(bz_0)\cdot\exp(2\pi i b_0 k + 2\pi i k\varepsilon)\\ &= \exp(bz_0)\cdot(\exp(2\pi i)^{b_0 k})\cdot\exp(2\pi i k\varepsilon)\\ &= \exp(bz_0)\exp(2\pi i k\varepsilon).\end{aligned}$$

Now take the case $k = 1$. Then $0 < 2\pi\varepsilon < 2\pi$, and so $\exp(2\pi i\varepsilon) \neq 1$. Therefore the function is not single-valued.

Finally, we consider the case where b is rational; say $b = \frac{m}{n}$ in lowest terms. Then it turns out there are n possible values for a^b. To see, this, in the above calculation we have $\varepsilon = \frac{m}{n}$, where we may assume $0 < m < n$. One can show that $\exp(2\pi i k_1 \frac{m}{n}) = \exp(2\pi i k_2 \frac{m}{n})$ if and only if $k_1 \equiv k_2 \pmod{n}$; the details are omitted.

Incidentally, if you take the case $b = \frac{1}{2}$, this explains why every nonzero number has two square roots! Over $\mathbb{R}$, we privilege positive square roots; but over $\mathbb{C}$, we can do no such thing. Finally, note that in the rational case, the n roots have the same magnitude, but their arguments differ by integer multiples of $2\pi/n$; in other words, they are evenly spaced around the origin.