HW 4 Selected Solutions

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3.3 Certainly if $\alpha = 0$, then the equation does not give a circle, and gives a line if and only if at least one of $\beta, \gamma \neq 0$. One checks that for $\alpha = 0$, the inequality gives exactly this.

Now assume $\alpha \neq 0$. We may rewrite equation (3.1) as

$$x^2 + \frac{\beta}{\alpha}x + y^2 + \frac{\gamma}{\alpha}y = -\frac{\delta}{\alpha}.$$

We complete the square to obtain

$$\left(x + \frac{\beta}{2\alpha}\right)^2 + \left(y + \frac{\gamma}{2\alpha}\right)^2 = \frac{\beta^2}{4\alpha^2} + \frac{\gamma^2}{4\alpha^2} - \frac{\delta}{\alpha} = \frac{\beta^2 + \gamma^2 - 4\alpha\delta}{4\alpha^2}.$$

This gives a circle if and only if the radius is positive; in other words, the right hand side must be > 0. Notice that $4\alpha^2 > 0$, so the right side is positive if and only if its numerator is positive, yielding the desired condition.

3.5 This can be done directly: that is, let $T(z) = \frac{az+b}{cz+d}$, write $T(z) = z$, then solve for z. We end up with a quadratic in z, which has at most 2 roots. (Technically you can also get the zero polynomial, but this can only occur if $T = \mathrm{id}$.) But the easier method is to use Corollary 3.13. Suppose T is a Möbius transformation with 3 fixed points, z_1, z_2, z_3. For each i, set $w_i = z_i$. Then by Corollary 3.13, T is the unique Möbius transformation with the property that $T(z_1) = w_1$, $T(z_2) = w_2$, $T(z_3) = w_3$. But the identity map *also* has this property! Therefore T must itself be the identity.

3.8 I believe the intended solution is to take a Möbius transformation T which sends S^1 to the real line (for example, $T(z) = \frac{i+iz}{1-z}$), and consider the composition $T \circ f$. Then we apply Exercise 2.20 to $T \circ f$. We have to be a little bit careful, though, since T sends 1 to ∞. So let $G_1 = \{z \in G : f(z) \neq 1\}$. Then $T \circ f$ is holomorphic on G_1 by Prop. 2.10f. Since its image lies in the real line, $T \circ f$ is a constant by Exercise 2.20. As T is a bijection, f must be constant on G_1. By continuity, f must be constant on G as well (I omit the details).

Here is a different method that avoids the subtleties in the above proof. The constant function 1 is holomorphic. Additionally, f is never 0 (since $0 \notin S^1$), and so $\frac{1}{f}$ is defined and holomorphic on G by Prop. 2.10c. As $f(z) \in S^1$ for all $z \in G$, we have $|f(z)|^2 = 1$. Therefore for all $z \in G$,

$$\frac{1}{f(z)} = \frac{|f(z)|^2}{f(z)} = \frac{f(z)\overline{f(z)}}{f(z)} = \overline{f(z)}.$$

It follows that both f and $\overline{f}$ are holomorphic on G, and so by Exercise 2.21, f is constant.

3.14 For (a), we have

$$[z, 1, 2, 3] = \frac{(z-1)(2-3)}{(z-3)(2-1)} = \frac{-z+1}{z-3}.$$

For (b), we want

$$[z, 1, 1+i, 2] = \frac{(z-1)(1+i-2)}{(z-2)(1+i-1)} = \frac{i-1}{i}\frac{z-1}{z-2} = (1+i)\frac{z-1}{z-2}.$$

For (c), we want the inverse of $[z, i, 1, -i]$. We have

$$[z, i, 1, -i] = \frac{(z-i)(1-(-i)}{(z-(-i))(1-i)} = \frac{1+i}{1-i}\frac{z-i}{z+i} = \frac{iz+1}{z+i}.$$

By Exercise 3.1, the inverse is

$$\frac{iz-1}{-z+i}.$$