HW 3 Selected Solutions

Prof. Shahed Sharif

2.6 Consider a line through the origin. Let $(a, b) \neq (0, 0)$ be any point on that line. Then the line is parametrized by $(x, y) = (at, bt)$. Along that line, the limit as $z \to 0$ of f is given by

$$\lim_{t\to 0} \frac{(at)^2 bt}{(at)^4 + (bt)^2} = \lim_{t\to 0} \frac{a^2 bt^3}{t^2(a^4t^2 + b)} = \lim_{t\to 0} \frac{a^2 b}{a^4 t^2 + b} t.$$

One sees that the limit is 0, and therefore the limit along any line through the origin is 0 as $z \to 0$.

Now consider the parabola. It is parametrized by $(x, y) = (t, t^2)$. The corresponding limit is

$$\lim_{t\to 0} \frac{t^2 t^2}{t^4 + t^4} = \frac{1}{2}.$$

Since this value is not 0, by Prop 2.2, the limit does not exist.

2.19 According to the definition of f, the value is 0 along the real an imaginary axes and 1 elsewhere. Since f is constant along the axes, at 0, we must have $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$. In particular, u_x, u_y, v_x, v_y are all zero at $z = 0$, and so the Cauchy-Riemann equations are trivially satisfied. But Theorem 2.13b requires the partials f to be defined and continuous at the origin, which it is not: choose any $t > 0$. Then the function $g(y) = f(t + iy)$ is not continuous at $y = 0$, and so its derivative does not exist there. This means $\frac{\partial f}{\partial y}$ is not defined at $(t, 0)$. As this holds for arbitrarily small positive t, the hypotheses of Theorem 2.13b are not satisfied.

2.21 By Prop. 2.10a, $(f + \bar{f})/2$ is also differentiable on G, and hence holomorphic. But $(f + \bar{f})/2 = \mathrm{Re}(f)$ is real, and so by exercise 2.20, $\mathrm{Re}(f)$ is constant. A similar argument applies to $(f - \bar{f})/2i = \mathrm{Im}(f)$, so $\mathrm{Im}(f)$ is constant. Since both $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are constant, f is constant.