HW 2 Selected Solutions

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1.31 Let our regions be G and H, and let $K = G \cup H$. We wish to show that K is open and connected. By problem A., K is open. To simplify the proof that K is connected, I will first prove the following:

Claim. *If X is a connected set, U, V are disjoint open sets, and $X \subset U \cup V$, then $X \subset U$ or $X \subset V$.*

Proof. If the conclusion does not hold, let $A = X \cap U$ and $B = X \cap V$; observe that neither is empty. Since $A \subset U$ and $B \subset V$, A and B are separated, but as $A \cup B = X$, this is a contradiction. The claim follows. □

Now suppose $K = A \cup B$ and we have disjoint open sets U, V with $A \subset U$, $B \subset V$. Take $z_0 \in G \cap H$. Without loss of generality, $z_0 \in A$. As $z_0 \in U$, we have $G \cap U \neq \varnothing$. From the claim, it follows that $G \subset U$. But by similar reasoning, $H \subset U$. Therefore $G \cup H = K \subset U$, and so B must be empty. It follows that K is connected.

Since K is both open and connected, it is a region.

1.32 Suppose $A \subset B$ and B is closed. let $x \in \partial A$. If $x \in B$, then there is nothing to show; so suppose $x \notin B$. I claim that $x \in \partial B$. For let $\varepsilon > 0$, and consider $D[x, \varepsilon]$. As $x \in \partial A$, $\exists y \in A \cap D[x, \varepsilon]$, and hence $y \in B \cap D[x, \varepsilon]$. On the other hand, $x \in D[x, \varepsilon]$ and $x \notin B$. As this holds $\forall \varepsilon > 0$, we have $x \in \partial B$. It follows that $\partial A \subset B \cup \partial B$. But B is closed, so $\partial B \subset B$, and hence $\partial A \subset B$.

Now suppose $A \subset B$ and A is open. Let $x \in A$. Since A is open, $\exists \varepsilon > 0$ such that $D[x, \varepsilon] \subset A$. But $A \subset B$, so $D[x, \varepsilon] \subset B$. Therefore x is in the interior of B.

1.33 Answers are not unique, but for example, for (a) we could do $\gamma(t) = 1 + i + e^{it}$ for $0 \leq t \leq 2\pi$, and for (d) we could do

$$\gamma(t) = \begin{cases} (t+5) - 2i & -6 \leq t \leq -4 \\ 1 + (t+2)i & -4 \leq t \leq 0 \\ (1-t) + 2i & 0 \leq t \leq 2 \\ -1 - (4-t)i & 2 \leq t \leq 6. \end{cases}$$

A. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary collection of open sets, and let $U = \cup_\lambda U_\lambda$. Let $x \in U$. Then $x \in U_\lambda$ for some λ. As U_λ is open, $\exists \varepsilon > 0$ such that $D[x, \varepsilon] \subset U_\lambda$. But $U_\lambda \subset U$, and hence $D[x, \varepsilon] \subset U$. This holds for all $x \in U$, and hence U is open.

B. Let U, V be open sets, and let $W = U \cap V$. Let $z_0 \in W$. Since $z_0 \in U$ and U is open, $\exists r > 0$ such that $D[z_0, r] \subset U$. Similarly, $\exists s > 0$ such that $D[z_0, s] \subset V$. Let $R = \min(r, s)$. Observe that $R > 0$ and

$$\begin{aligned} &D[z_0, R] \subset D[z_0, r], \text{ so } D[z_0, R] \subset U, \text{ and} \\ &D[z_0, R] \subset D[z_0, s], \text{ so } D[z_0, R] \subset V. \end{aligned}$$

Therefore $D[z_0, R] \subset W$. Since $z_0 \in W$ was arbitrary, this shows that W is open.

C. Take for instance the sets $D[0, r]$ for $r > 0$. The intersection of these is just $z = 0$, which is not open since any open disk around 0 must contain at least one other point (and in fact infinitely many other points!).

D. Let B be a closed set, and set $A = B^c$. Take $x \in A$. I claim that $\exists \varepsilon > 0$ such that $D[x, \varepsilon] \subset A$. If not, then for all ε, $\exists y \in D[x, \varepsilon] \cap B$. But as $x \in D[x, \varepsilon]$ and $x \notin B$, this would imply that $x \in \partial B$. But B is closed, and so $\partial B \subset B$, implying that $x \in B$. This contradicts $x \in A$. Thus there is such an ε, proving that A is open.