HW 11 Selected Solutions

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7.32 Suppose $\lim |\frac{c_{k+1}}{c_k}| = 0$. Let $r = |z - z_0|$, and let $\varepsilon = \frac{1}{2r}$. Then $\exists N$ such that $n \geq N$ implies $|\frac{c_{n+1}}{c_n}| < \varepsilon$, or in other words

$$|c_{n+1}| < |c_n|/2r.$$

A straightforward induction implies that $|c_{N+k}| < \frac{|c_N|}{2^k r^k}$, and so

$$|c_{N+k}(z - z_0)^{N+k}| < \frac{|c_N|}{2^k}.$$

Let $M_n = |c_n(z - z_0)^n|$ for $n < N$ and, for $n \geq N$, letting $k = N - n$, set $M_n = \frac{|c_N}{2^k}$. The series $\sum M_n$ converges since it is the sum of a finite series (the terms up to $n = N - 1$) plus a geometric series (the remaining terms) with common ratio $\frac{1}{2}$. By the Comparison Test, our original series converges absolutely.

Suppose $\lim |\frac{c_k}{c_{k+1}}| = R$. The argument is similar. Let $r = |z - z_0|$, and assume that $r < R$. Let s be any value strictly between r and R; say, $s = \frac{1}{2}(r + R)$. There exists N such that $n \geq N$ implies that $|\frac{c_k}{c_{k+1}} - R| < (R - s)$, and so in particular $|\frac{c_k}{c_{k+1}}| \geq s$. A straighforward induction shows that $|c_{N+k}| \leq |c_N|/s^k$. Then

$$\begin{aligned} |c_{N+k}(z - z_0)^{N+k}| &\leq |c_N| r^{N+k}/s^k \\ &= r^N |c_N| \left(\frac{r}{s}\right)^k. \end{aligned}$$

Since $0 < r/s < 1$, the series $\sum_{k=0} r^N |c_N| (r/s)^k$ converges. By the Comparison Test, $\sum_{k=0} |c_{N+k}(z - z_0)^{N+k}|$ converges absolutely. The full series therefore also converges absolutely since ignoring the first N terms does not affect convergence.

A similar argument shows divergence outside the disk: if $|z - z_0| > R$, let $r = |z - z_0|$, choose s with $R < s < r$, check that $|\frac{c_k}{c_{k+1}}| \leq s$, and hence $|c_{N+k}| \geq |c_N|/s^k$. Then

$$\begin{aligned} |c_{N+k}(z - z_0)^{N+k}| &\geq |c_N| r^{N+k}/s^k \\ &= r^N |c_N| \left(\frac{r}{s}\right)^k, \end{aligned}$$

and now since $r/s > 1$, the geometric series diverges. The details are similar.

7.33f If $|z| < 1$, then $|(\cos k) z^k| < |z|^k$. But $\sum |z|^k$ is a convergent geometric series, so by the Comparison Test, $\sum (\cos k) z^k$ converges absolutely. Thus the radius is at least 1.

Now suppose $r > 1$. I claim that the series diverges when $z = r$. The key fact is that as k varies in $\mathbb{Z}$, the values of $\cos k$ are dense in $[-1, 1]$. In particular, there are infinitely many values of k such that $\cos k > 0.9$, and so for those values of k, $(\cos k)r^k > 0.9$. That means $\lim_{k\to\infty}(\cos k)r^k \neq 0$. By the Test for Divergence, the series diverges. Therefore the radius of convergence is at most 1.

We conclude that the radius is exactly 1.

8.7 Let $z_0 \in G$. Since G is open, $\exists r > 0$ such that $D[z_0, r] \subset G$. Let γ be any piecewise smooth closed path in $D[z_0, r]$. Since $D[z_0, r]$ is simply connected, γ is contractible in $D[z_0, r]$, and hence $\int_\gamma f_n \, dz = 0$. But since $f_n \to f$ uniformly on G, $\int_\gamma f_n \, dz$ converges to $\int_\gamma f \, dz$, and so the latter integral is zero. We also know that f is continuous (it is a uniform limit of continuous functios), so by Morera's Theorem, f is holomorphic at z_0. This holds for all $z_0 \in G$, and hence f is holomorphic on G.

Note that you cannot just apply Morera's Theorem on G, because the hypothesis of Morera's Theorem might not be true! Take for example $G = \mathbb{C} - \{0\}$, $f_n(z) = f(z) = \frac{1}{z}$, and $\gamma = C[0, 1]$. Then $\int_\gamma f_n \, dz = 2\pi i$. The issue here is invoking Cauchy's Theorem on f_n; you can only use it if the path is contractible in G.