HW 10 Selected Solutions

Prof. Shahed Sharif

7.8 Suppose for the sake of contradiction that there is a limit point L and two accumulation points p_1, p_2. Let $d = \max(|p_1 - L|, |p_2 - L|)$. It is possible that $L = p_1$ or $L = p_2$, but since $p_1 \neq p_2$, we cannot have both equalities. Therefore $d > 0$. Without loss of generality, $|p_1 - L| = d$. Let $\varepsilon = d/2$. Since $\lim_{n\to\infty} a_n = L$, $\exists N$ such that $n \geq N$ implies that $|a_n - L| < \varepsilon$. But p_1 is an accumulation point, so for the same N, $\exists m \geq N$ such that $|a_m - p_1| < \varepsilon$. Thus

$$|a_m - p_1| < \varepsilon \text{ and } |a_m - L| < \varepsilon.$$

We now derive a contradiction. Using the triangle inequality and the above inequalities, we have

$$\begin{aligned} d &= |p_1 - L| \\ &\leq |p_1 - a_m| + |a_m - L| \\ &< \varepsilon + \varepsilon \\ &= \frac{d}{2} + \frac{d}{2}, \end{aligned}$$

implying that $d < d$. This is a contradiction, and so the claim is proved.

7.21 If $x \neq \pi/2$, then $|\sin(x)| < 1$, and hence

$$\lim_{n\to\infty} \sin^n(x) = 0.$$

As $\sin(\pi/2) = 1$, we have

$$\lim_{n\to\infty} \sin^n(\pi/2) = \lim_{n\to\infty} 1^n = 1.$$

This shows pointwise convergence. If the convergence were uniform, then since $\sin^n(x)$ is continuous for all n, we would conclude that $f(x)$ is continuous by Prop 7.25. But it clearly isn't continuous, so the convergence is not uniform.

7.29c The denominator is the tricky part! Let $d = 1 - r$, so $0 < d < 1$. Then since $|z| \leq r < 1$ on our domain, $|z|^k \leq r^k < r$, and so

$$|1 + z^k| \geq 1 - |z|^k \geq 1 - r = d$$

for all $k \geq 0$. We therefore get that

$$\left|\frac{z^k}{z^k + 1}\right| \leq \frac{r^k}{d}.$$

The series $\sum \frac{r^k}{d}$ is geometric and converges to

$$\frac{1}{d(1 - r)}.$$

By the Weierstrass M-test, the original series converges uniformly on $\overline{D}[0, r]$.