Name:

Math 536: Final exam

December 13, 2023

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. Avoid using the back of the page; instead, there is an extra sheet at the end that you can use. You may use any result proven in the text or in lecture, but not homework problems. Standard algorithms require no justification. Calculators are not allowed.

1. No justification is necessary for this problem, but you may provide justification to receive partial credit.

   (a) (5 points) Convert $2e^{3\pi i/4}$ to rectangular form.

   (a) ______________

   Solution: $-\sqrt{2} + i\sqrt{2}$

   (b) (5 points) Compute $\displaystyle\int_{C[0,1]} \frac{z^2+1}{z}\, dz$.

   (b) ______________

   Solution: By Cauchy Integral Formula, this is $2\pi i(z^2+1)|_{z=0} = 2\pi i$.

   (c) (5 points) The function $\dfrac{\cos(z)-1}{\sin(z)^n}$ has a removable singularity at $z=0$. Find the largest possible value of n.

   (c) ______________

   Solution: $n = 2$: $\cos z - 1 = -\frac{z^2}{2} + h.o.t.$, $\sin z = z(1 - \frac{z^3}{6} + h.o.t)$, so

   $$\sin(z)^n = z^n(1 - h.o.t.).$$

   Thus the numerator has a zero of order 2, and the denominator a zero of order n. The be removable, we need $2 - n \geq 0$.

   (d) (5 points) Give an example of a sequence of functions $(f_n(z))$ which converge on $D[0,2]$, but not uniformly.

   Solution: $f_n(z) = (\frac{z}{2})^n$

   (e) (5 points) Calculate $\mathrm{Res}_{z=0} \dfrac{\sin z}{z^4}$

   (e) ______________

Solution:

$$\begin{aligned}\frac{\sin z}{z^4} &= \frac{z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots}{z^4} \\ &= z^{-3} - \frac{1}{6}z^{-1} + \frac{1}{120}z + \cdots\end{aligned}$$

Thus the residue is $-\frac{1}{6}$.

2. (a) (10 points) Compute i^i.

(a) ____________

Solution:

$$\begin{aligned} i^i &= e^{\log i^i} \\ &= e^{i \log i} \\ &= e^{i(i\pi/2)} \\ &= e^{-\pi/2}. \end{aligned}$$

Note that you get a different answer if you use a different branch of the logarithm, but complex exponentials are defined using the principal branch.

(b) (10 points) Compute $\displaystyle\int_{C[1,2]} \frac{\exp(z^2)}{(z-1)^2}\, dz$.

(b) ____________

Solution: By Cauchy Integral Formula, this is

$$2\pi i \left[\frac{d}{dz} e^{z^2}\right]_{z=1} = 2\pi i \left[2ze^{z^2}\right]_{z=1} = 4\pi e i.$$

(c) (10 points) Suppose $\sum_{k=0}^{\infty} c_k (z-i)^k$ is a power series for $\frac{1}{1-z}$. What is its radius of convergence?

(c) ____________

Solution: As large as possible, which means the largest disk that does not contain 1. The distance between i and 1 is $|i-1| = \sqrt{2}$, and so that is the radius.

(d) (10 points) Prove or give a counterexample: if $f(z)$ is holomorphic in a region G and γ is a piecewise-smooth closed path in G, then $\int_\gamma f\, dz = 0$.

Solution: It is false: let $G = \mathbb{C} - \{0\}$, $f(z) = \frac{1}{z}$, $\gamma = C[0,1]$. Then $\int_\gamma f\, dz = 2\pi i$. Note also that 1b and 2b give counterexamples for the appropriate choice of G.

3. (15 points) Let G be a region, and (f_n) a sequence of holomorphic functions on G which converge uniformly to f. Prove that f is holomorphic.

Solution: This was a homework problem; see those solutions.

4. (15 points) Show that the intersection of two open sets is open.

Solution: This was a homework problem; see those solutions.

5. (15 points) Show that $\displaystyle\sum_{k=0}^{\infty} \frac{z^k}{z^k+1}$ converges uniformly on $\overline{D}[0, \frac{1}{2}]$.

Solution: This was a homework problem; see those solutions.

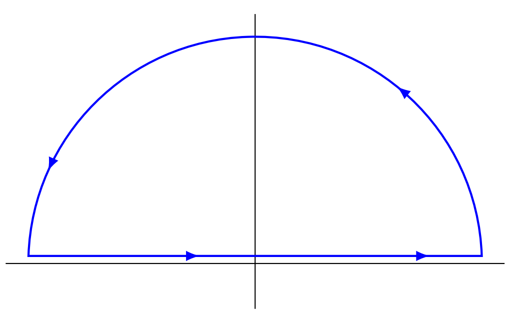
6. (15 points) Suppose u is a harmonic function on the open unit disk centered at the origin. Show that all partial derivatives of u, to any order, are harmonic.

Solution: This was a homework problem; see those solutions.

7. (20 points) Compute

$$\int_{-\infty}^{\infty} \frac{e^{i\pi x}}{x^2 - 2x + 2} \, dx.$$

Solution: Let $f(z) = e^{i\pi z}/(z^2 - 2z + 2)$. Let $R > \sqrt{2}$ (we'll see why in a moment), and let γ_R be the positively-oriented boundary of the upper semicircle centered at the origin with radius R. Let ρ_R be the part of the boundary along the real axis, and σ_R the rest of the boundary; that is, the circular arc.



We will compute $\displaystyle\lim_{R\to\infty} \int_{\gamma_R} f(z)\, dz$ in two different ways.

The first way is using the Residue Theorem (or you can use the Cauchy Integral Formula). The denominator $x^2 - 2x + 2 = (x - (1+i))(x - (1-i))$ has two roots: $1 + i$ and $1 - i$.

Thus the only pole of f inside γ_R is $1+i$ (this is why we need $R > \sqrt{2}$). The pole is simple, so the residue is

$$\frac{e^{i\pi(1+i)}}{(1+i)-(1-i)} = \frac{e^{i\pi}e^{i^2\pi}}{2i} = -\frac{e^{-\pi}}{2i}.$$

By the Residue Theorem,

$$\int_{\gamma_R} f(z)\,dz = -2\pi i\frac{e^{-\pi}}{2i} = -\pi e^{-\pi}.$$

Note that the value does not depend on R, so in particular the limit as $R \to \infty$ of the integral is also $-\pi e^{-\pi}$.

Now we evaluate it a different way. We have

$$\int_{\gamma_R} f\,dz = \int_{\rho_R} f\,dz + \int_{\sigma_R} f\,dz.$$

As $R \to \infty$, the ρ_R part becomes precisely the integral given in the problem. For the σ_R part, observe that on σ_R, with $z = x+iy$ and R large, we have

$$\begin{aligned}|f(z)| &= \left|\frac{e^{i\pi z}|}{z^2-2z+2}\right| \\ &= \frac{|e^{i\pi x + i^2\pi y}|}{|z^2-2z+2|} \\ &\le \frac{|e^{i\pi x}e^{-\pi y}|}{R^2-R-2} \\ &\le \frac{1}{R^2-R-2}.\end{aligned}$$

Thus

$$\left|\int_{\sigma_R} f(z)\,dz\right| \le \frac{\pi R}{R^2-R-2}.$$

As $R \to \infty$, the right side goes to zero, and hence so does the left side. It follows that

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{i\pi x}}{x^2-2x+2}\,dx &= \lim_{R\to\infty}\int_{\rho_R} f(z)\,dz \\ &= \lim_{R\to\infty}\int_{\gamma_R} f(z)\,dz \\ &= -\pi e^{-\pi}.\end{aligned}$$