Name:

Math 536: Exam 1

October 18, 2023

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. Avoid using the back of the page; instead, there is an extra sheet at the end that you can use. You may use any result proven in the text or in lecture through §4.2, but not homework problems. Standard algorithms require no justification. Calculators are not allowed.

1. (a) (5 points) Compute $(1-2i)^2$.

   Solution: This is $-3-4i$.

   (b) (5 points) Write $-3+\sqrt{3}i$ in polar form.

   Solution: This is $2\sqrt{3}e^{5\pi i/6}$.

   (c) (5 points) Find a Möbius transformation T satisfying $T(0)=i, T(1)=-1, T(\infty)=-i$.

   Solution: Let $S=[z,i,-1,-i]$. We have

   $$S=\frac{(z-i)(-1+i)}{(z+i)(-1-i)}=\frac{-iz-1}{z+i}.$$

   But

   $$T(z)=S^{-1}(z)=\frac{iz+1}{-z-i}.$$

   (d) (5 points) Given an example of a function $f:\mathbb{C}\to\mathbb{C}$ such that the partials $\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$ exist and are continuous everywhere, but f is not differentiable anywhere.

   Solution: $f(z)=\overline{z}$; we showed this in class.

2. (a) (10 points) Show that if f is an entire function and $f(z)$ is real for all z, then f is constant.

   Solution: This was a homework problem.

   (b) (10 points) Show that if f is an entire function and $|f(z)|=1$ for all z, then f is constant.

   Solution: Ditto.

3. (15 points) Show that the union of two closed sets is closed.

Solution: Let A, B be closed sets, and let $C = A \cup B$. Let $x \in \partial C$. If $x \in A$ or $x \in B$, then certainly $x \in C$. So suppose $x \notin A$, $x \notin B$. Since $x \in \partial C$, for each $\varepsilon > 0$, $\exists c \in C$ such that $c \in D[x, \varepsilon]$. Set $\varepsilon = \frac{1}{n}$ for each n, and let c_n be the corresponding element of C. We have $c_n \in A$ or $c_n \in B$ for each n. Since there are infinitely many n, there is an infinite subsequence of c_n which lies entirely in A or entirely in B; without loss of generality we'll say this subsequence lies in A. Then given $\varepsilon > 0$, there is n coming from this subsequence for which $\frac{1}{n} < \varepsilon$, and hence $c_n \in A \cap D[x, \varepsilon]$. This holds for all ε, and so $x \in \partial A$. But A is closed, so $x \in A$, and hence $x \in C$. It follows that C is closed.

Here is an alternative proof: Let A, B be closed set and $C = A \cup B$. We will show the contrapositive of the definition of closed: if $x \notin C$, then $x \notin \partial C$. As $x \notin A$, we have $x \notin \partial A$, so $\exists r_1$ such that $D[x, r_1] \cap A = \varnothing$. Similarly, $x \notin B$, so $\exists r_2$ such that $D[x, r_2] \cap B = \varnothing$. Let $r = \min(r_1, r_2)$. Then $D[x, r]$ is disjoint from both A and B, and hence $D[x, r] \cap C = \varnothing$. It follows that $x \notin \partial C$, as desired.

4. (15 points) Show that the function $f(z) = z^2$ is continuous on all of $\mathbb{C}$, using the definition of continuity.

Solution: Choose $w \in \mathbb{C}$. I wish to show that $\lim_{z \to w} z^2 = w^2$. Take $\varepsilon > 0$. Choose $\delta = \min(\frac{\varepsilon}{2|w|+1}, 1)$, and suppose $|z - w| < \delta$. Then

$$\begin{aligned} |z^2 - w^2| &= |z + w||z - w| \\ &< \delta |z + w| \\ &\leq \delta(|z| + |w|) \\ &\leq \delta(|w| + 1 + |w|) \\ &= \frac{\varepsilon}{2|w| + 1}(2|w| + 1) \\ &= \varepsilon. \end{aligned}$$

This holds for all $\varepsilon > 0$, and so continuity follows.

5. (15 points) Let $i\mathbb{R}$ denote the imaginary axis and S^1 the unit circle. Find a Möbius transformation T satisfying $T(i\mathbb{R}) \subset S^1$; make sure to justify your answer!

Solution: We can do this by sending any 3 distinct points on $i\mathbb{R}$ to S^1; since Möbius transformations send generalized circles to generalized circles, it would follow that $i\mathbb{R}$ gets sent to S^1. On $\hat{C}$, ∞ lies on every line, so we can even let ∞ be one of the points.

There are thus many answers; but for instance let $S_1(z) = [z, -i, 0, i]$ and $S_2(z) = [z, 1, i, -1]$. Then we know that $S_2^{-1} \circ S_1$ send $-i, 0, i$ to $1, i, -1$, respectively. Thus $T = S_2^{-1} \circ S_1$ works. We get $S_1 = \frac{z+i}{-z+i}$ and $S_2 = \frac{-iz+i}{z+1}$, from which one can obtain

$$T(z) = \frac{z+1}{iz - i}.$$

6. (15 points) Let $c \in \mathbb{C}$, $c \neq 0$, and let $f(z) = z^c$. Prove that $f'(z) = cz^{c-1}$.

Solution: We write $f(z) = \exp(c \operatorname{Log}(z))$. Then

$$\begin{aligned} f'(z) &= c \frac{d}{dz} \operatorname{Log}(z) \cdot \exp(c \operatorname{Log}(z)) \\ &= \frac{c}{z} \exp(c \operatorname{Log}(z)) \\ &= \frac{c}{z} z^c \\ &= c z^{c-1}. \end{aligned}$$

Note that even though Log is not holomorphic on $\mathbb{C} - \{0\}$, it is differentiable on that domain.

7. Let γ be the polygonal path from $2i$ to $-1+i$ to $-1-i$ to $-2i$.

(a) (10 points) Compute $\int_\gamma \exp(\pi z)\, dz$.

Solution: The function $\frac{1}{\pi}\exp(\pi z)$ is an antiderivative, and so applying the corresponding theorem yields

$$\frac{1}{\pi}\exp(-2i\pi) - \frac{1}{\pi}\exp(2i\pi) = 0.$$

(b) (10 points) Compute $\int_\gamma \frac{1}{z}\, dz$.

Solution: The principal branch of logarithm is *not* an antiderivative along γ since it is discontinuous at -1. However, we can use a different branch of log; namely, the one with $\log(z) = \ln|z| + i\arg(z)$ with $0 \leq \arg(z) < 2\pi$. This branch is continuous on $\mathbb{C} - \mathbb{R}^{>0}$. Now applying our theorem, the integral is $\log(-2i) - \log(2i) = (\ln|2| + i\frac{3\pi}{2}) - (\ln|2| + i\frac{\pi}{2}) = i\pi$.