

Supplemental Notes for Hoffman-Kunze

Prof. Shahed Sharif

1 Linear equations

1.6 Invertible matrices

Here's an outline of the results of this section. I give some comments on some of the proofs. For those that I do not comment on, you should check your notes, the text, or do it as an exercise.

- If a matrix has a left inverse and a right inverse, then they are the same.
- The inverse of a matrix is invertible.
- Products of matrices are invertible.
- An elementary matrix is invertible. This is because the elementary row operation is also invertible. Applying both row operations in sequence to the identity gives the product of the corresponding elementary matrices. This product is I , so they are inverses.
- If A is invertible, then it is row equivalent to the identity. Well, A is row equivalent to its rref R , which is either the identity or has a zero row. If it has a zero row, then RB will have a zero row no matter what B we pick, so R is not invertible. But $R = E_1 \cdots E_t A$ for some elementary matrices, so now use the fact that products of invertibles are invertible.
- Invertibles are products of elementary matrices.
- The relationship between invertibility and the solutions of $Ax = 0$; namely, A is invertible if and only if $Ax = 0$ has a unique solution. This follows from replacing A with its rref.
- If A has a one-sided inverse, then it has a two-sided inverse. Use the equation $Ax = 0$ and the previous statement.

2 Vector spaces

2.3 Bases and Dimension

Lemma 2.1. Suppose A is an $m \times n$ matrix and c is a $n \times 1$ matrix (a column vector). Then Ac is a linear combination of the columns of A , with the j th coefficient equal to the j th entry of c .

Proof. This is just a tedious algebra problem. Let A_j be the j th column of A , so

$$A_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}.$$

Let the entries of c be $c_1, \dots, c_n$, and $v = \sum_{j=1}^n c_j A_j$. I wish to show that $v = Ac$. The sizes match up, so let us compare the entries. The i th entry of v is the sum of the i th entries of $c_j A_j$, which equals

$$\sum_{j=1}^n c_j a_{ij}.$$

But from our matrix multiplication formula, this equals the i th entry of Ac , completing the proof. $\square$

2.5 Summary of Row-Equivalence

I fill in the gap from the proof in class; namely, if A and B have the same row space, then they are row equivalent. First, a lemma.

Lemma 2.2. *Suppose W is a finite dimensional vector space over F , and let $\mathcal{B} = (\alpha_1, \dots, \alpha_n)$, $\mathcal{B}' = (\alpha'_1, \dots, \alpha'_n)$ be two bases for W . Define P_{ij} by*

$$\alpha'_i = \sum_{j=1}^n P_{ij} \alpha_j.$$

Then $P \in GL_n(F)$.

Proof. This follows from the proof of the change-of-basis theorem; that is, the proof is the same, since the matrix is the same. $\square$

Proposition 2.3. *Suppose $A, B \in M_{m \times n}(F)$ have the same row space. Then A and B are row-equivalent.*

Proof. Note that we may replace A or B by a row-equivalent matrices, since row-equivalence is transitive and also preserves the row space. So we may assume that both A and B are in rref. One can show that the nonzero rows of A and B each form bases for the row space; this is a good exercise, but it is also done in the text. Since the row spaces are the same, and two bases must have the same number of elements, the number of nonzero rows of A is the same as that of B ; say that this number is d . Let the nonzero rows of A be $\alpha_1, \dots, \alpha_d$, and for B , $\alpha'_1, \dots, \alpha'_d$. Let $P \in GL_d(F)$ be as in the lemma. Let I be the $(m-d) \times (m-d)$ identity matrix, and take Q to be the block matrix

$$\left[\begin{array}{c|c} P & 0 \\ \hline 0 & I \end{array} \right]$$

We have that $QA = B$; I leave this as an exercise. (My suggestion would be to prove the equality one row at a time.) Additionally, since P is invertible, it is row-equivalent to the $d \times d$ identity. Therefore Q is row-equivalent to the $m \times m$ identity, and hence is also invertible.

Next, Q can be written as a product of elementary matrices $E_1 \cdot E_2 \cdots E_t$ since it is invertible, so

$$B = E_1 \cdot E_2 \cdots E_t A.$$

Each elementary matrix accomplishes an elementary row operation, and hence we have proved the claim. $\square$

Let A be an $m \times n$ matrix and let

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

where we view the b_i s as unknowns. Consider the augmented matrix $[A \mid b]$. Write the rref as $[R \mid L]$; note that the *entire* matrix is in rref, not just the left part. For each row of R which is a zero row, consider the corresponding entry in the right-hand column L ; call these entries $L_1, \dots, L_t$. (The notation is funny: L_1 need not be the *first* entry of L .) Note that since $[R \mid L]$ is row-equivalent to $[A \mid b]$, each L_i is a linear combination of the b_j .

Proposition 2.4. *Given $b \in F^m$, b lies in the span of the columns of A if and only if $L_i(b) = 0$ for all i , $1 \leq i \leq t$.*

Proof. Let $v_1, \dots, v_n$ be the columns of A , and let W be their span.

Suppose that b lies in W . Then $\exists c_1, \dots, c_n$ such that $\sum c_i v_i = b$. Letting $c = (c_1, \dots, c_n)$, we know this means that

$$Ac = b.$$

Therefore the system $Ax = b$ has a solution. To have a solution, every zero row of R must have the corresponding entry in the column L equal 0. Therefore $L_i(b) = 0$ for all i .

The argument is reversible: if the $L_i(b) = 0$ for all i , then $Rx = L(b)$ is consistent, so $Ax = b$ is consistent, and hence $\exists c$ such that $Ac = b$. Writing Ac as a linear combination of the columns of A , we see that $b \in W$. $\square$