

Homework 5

Prof. Shahed Sharif

3.3.6 If $\dim V = \dim W = n$, then we know $V \cong F^n$ and $W \cong F^n$. But being isomorphic is transitive, so $V \cong W$.

Now suppose $V \cong W$. Let $T : V \rightarrow W$ be an isomorphism. Let $\mathcal{B}$ be a basis for V of size $\dim V$. Then $\mathcal{B}$ is linearly independent. By Theorem 3.2.8, $T(\mathcal{B})$ is linearly independent in W , and so by Theorem 2.3.4, the length of $\mathcal{B}$ is $\leq \dim W$. Therefore $\dim V \leq \dim W$. A symmetric argument using T^{-1} gives the reverse inequality.

3.4.2 I only do (b). We have

$$\begin{aligned} T(\alpha_1) &= (1, -3) = -3 \cdot \beta_1 + 1 \cdot \beta_2 \\ T(\alpha_2) &= (2, 1) = 1 \cdot \beta_1 + 2 \cdot \beta_2 \\ T(\alpha_3) &= (1, -1) = -1 \cdot \beta_1 + 1 \cdot \beta_2. \end{aligned}$$

Therefore the matrix is

$$\begin{bmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}.$$

3.4.10 Let $d = \dim \ker S$. Since $\ker S$ is a subspace of $\mathbb{R}^2$, $d = 0, 1$, or 2 . If $d = 2$, then $S = 0$. If $d = 0$, then S is non-singular, hence invertible. Multiplying both sides of $S^2 = S$ by the inverse, we get $S = I$.

Now consider the case where $d = 1$. Let (β) be a basis for $\ker S$. By Rank-Nullity, $\text{Im } S$ is 1-dimensional, so $\exists \alpha \in V$ such that α forms a basis for $\text{Im } S$. Since $\alpha \in \text{Im } S$, $\exists \gamma \in V$ such that $S\gamma = \alpha$. Applying S to both sides, we get $S^2\gamma = S\alpha$. But $S^2 = S$, so $S^2\gamma = S\gamma = \alpha$. Therefore $S\alpha = \alpha$.

I claim that (α, β) is a basis for $\mathbb{R}^2$. For linear independence, suppose that $a\alpha + b\beta = 0$. Applying S , we get $a\alpha = 0$, and hence $a = 0$. From there, it follows that $b = 0$. This shows linear independence. Since our list has size 2, which is the dimension of $\mathbb{R}^2$, we have found a basis.

Now we compute the matrix of S with respect to (α, β) . As $S(\alpha) = \alpha$, the first column is $[1, 0]$. As $S\beta = 0$, the second column is $[0, 0]$. The claim follows.