

Homework 3

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2.4.6 For (a), suppose

$$c_1 \cdot 1 + c_2 e^{ix} + c_3 e^{-ix} = 0.$$

We plug in $x = \pi/2$, $x = 0$, and $x = \pi$ to obtain the three equations

$$c_1 + ic_2 - ic_3 = 0$$

$$c_1 + c_2 + c_3 = 0$$

$$c_1 - c_2 - c_3 = 0.$$

Adding the last two equations shows $c_1 = 0$. Then adding $-i$ times the first equation to the second shows that $c_2 = 0$. It follows that $c_3 = 0$ as well. Linear independence follows.

For (b), we have $e^{ix} = \cos x + i \sin x$ and $e^{-ix} = \cos x - i \sin x$. Therefore if we let

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & i & -i \end{bmatrix}$$

Then $f_j = \sum_i Q_{ij} g_i$. It follows that $P = Q^{-1}$; in this case, we get

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2}i \\ 0 & \frac{1}{2} & \frac{1}{2}i \end{bmatrix}$$

A. The algorithm is as follows:

1. Test L for linear dependence.
2. If there is a linear dependence equation, choose an $\alpha_j \in L$ for which the coefficient of α_j in the equation is nonzero.
3. Remove α_j from L .
4. Repeat the above steps until linear independence is achieved, and output the resulting list.

I claim that at each stage, the list L spans V . By induction, it suffices to show this for one iteration of the loop. Renaming the elements of L (since in the middle of the algorithm, we will have removed some of the elements of the original L), assume that we have

$$L = (\beta_1, \beta_2, \dots, \beta_m).$$

Suppose $\sum c_i \beta_i = 0$ with at least one c_i nonzero; without loss of generality, $c_m \neq 0$. Then

$$\beta_m = - \sum_{i=1}^{m-1} \frac{c_i}{c_m} \beta_i.$$

Let L' be L with β_m removed. I claim that $\text{Span}(L') = \text{Span}(L)$. The containment $\text{Span}(L') \subseteq \text{Span}(L)$ is obvious. For the other direction, given $v \in \text{Span}(L)$, we have $\exists a_1, \dots, a_m$ such that $v = \sum^m a_i \beta_i$. But then

$$\begin{aligned} v &= \sum_{i=1}^{m-1} a_i \beta_i + a_m \beta_m \\ &= \sum_{i=1}^{m-1} a_i \beta_i - a_m \sum_{i=1}^{m-1} \frac{c_i}{c_m} \beta_i \\ &= \sum_{i=1}^{m-1} \left(a_i - \frac{a_m c_i}{c_m} \right) \beta_i, \end{aligned}$$

and hence $v \in \text{Span}(L')$. Thus the span is preserved. Eventually the size of the spanning set will be $\leq \dim V$, so the algorithm terminates. Once it does, we have a linearly independent spanning set; that is, a basis.

B. There are

$$(2^n - 1)(2^n - 2)(2^n - 2^2) \cdots (2^n - 2^{n-1}).$$

First I show the following: if V is a dimension k vector space over $\mathbb{F}_2$, then $\#V = 2^k$. For let $\alpha_1, \dots, \alpha_k$ be a basis. Every vector can be uniquely written as $\sum c_i \alpha_i$ for some $c_i \in \mathbb{F}_2$. There are 2 choices for each c_i , and hence 2^k possible choices for the sequence of c_i . The claim follows.

Now, we recursively choose basis vectors. For the first vector, we can choose any nonzero vector α_1 ; there are clearly $2^n - 1$ such choices. Suppose we have chosen the first k vectors so that they are linearly independent; these span a subspace of dimension 2^k . We must choose the next vector v_{k+1} , to not lie in this subspace; this suffices by the Lemma on pg. 45 of the text (which annoyingly is not numbered). There are $2^n - 2^k$ such choices. We continue in this way until we have $n = \dim V$ vectors. By elementary combinatorics, our formula above is correct.

Note that if we work over $\mathbb{F}_q$ instead (the finite field with q elements), essentially the same argument works if we replace 2 with q .