

Homework 2

Prof. Shahed Sharif

1.6.6 Firstly, if $B = [0\ 0]$, then C is the zero matrix, and is clearly not invertible.

Suppose $B \neq 0$. Let $B = [b_1\ b_2]$. Let D be the column matrix $[b_2, -b_1]$. Note that $BD = [0]$. That means that $CD = (AB)D = A(BD) = 0$. If C were invertible, multiplying both sides on the left by C^{-1} would yield $D = 0$. But $B \neq 0$, so $D \neq 0$, yielding a contradiction. Therefore C is not invertible.

2.1.6 The set of all functions from $\mathbb{C}$ to $\mathbb{C}$ is a vector space, and we can just restrict to functions on $\mathbb{R}$, so V inherits these operations and their properties. Thus we need only show that V is nonempty and closed under vector addition and scalar multiplication.

Certainly $0 \in V$, since $\overline{0(t)} = 0 = (-0)(t)$. (I'm using 0 to mean both the scalar and function 0 .) Let $f, g \in V$ and $c, t \in \mathbb{R}$. Then

$$\begin{aligned}\overline{(f+g)(t)} &= \overline{f(t) + g(t)} \\ &= \overline{f(t)} + \overline{g(t)} \\ &= f(-t) + g(-t) \\ &= (f+g)(-t).\end{aligned}$$

Note that the third equality follows from the definition of V . Thus $f+g \in V$. We also have

$$\begin{aligned}\overline{(cf)(t)} &= \overline{cf(t)} \\ &= cf(-t) \\ &= (cf)(-t).\end{aligned}$$

Here, we are using the fact that $c \in \mathbb{R}$, so $\bar{c} = c$. The claim now follows.

The easiest example is $f(t) = it$.

2.2.8 For (a), if $f, g \in V_e$, then $f(-x) = f(x)$ and $g(-x) = g(x)$. But then

$$\begin{aligned}(f+g)(-x) &= f(-x) + g(-x) \\ &= f(x) + g(x) \\ &= (f+g)(x).\end{aligned}$$

Therefore $f+g$ is even. Additionally for c a scalar, $cf(-x) = cf(x)$, so cf is even. Finally, the zero function is even. Therefore V_e is a subspace.

The case of V_o is similar, and I omit it.

For (b), given $f \in V$, let $f_e(x) = \frac{1}{2}(f(x) + f(-x))$ and $f_o = \frac{1}{2}(f(x) - f(-x))$. We have

$$\begin{aligned} f_e(-x) &= \frac{1}{2}(f(-x) + f(x)) \\ &= \frac{1}{2}(f(x) + f(-x)) \\ &= f_e(x), \end{aligned}$$

and

$$\begin{aligned} f_o(-x) &= \frac{1}{2}(f(-x) - f(x)) \\ &= -\frac{1}{2}(f(x) - f(-x)) \\ &= -f_o(x), \end{aligned}$$

so $f_e \in V_e$ and $f_o \in V_o$. Finally,

$$\begin{aligned} f_e(x) + f_o(x) &= \frac{1}{2}(f(x) + f(-x) + f(x) - f(-x)) \\ &= \frac{1}{2} \cdot 2f(x) \\ &= f(x). \end{aligned}$$

This proves (b).

For (c), given $f \in V_e \cap V_o$, for any $x \in \mathbb{R}$, $f(x) = f(-x)$ and $f(x) = -f(-x)$. Thus $f(-x) = -f(-x)$. This can only occur if $f(-x) = 0$, and hence $f(x) = 0$. As this holds for all x , we must have that f is identically zero. We already showed that the zero function is in both spaces, and so the equality follows.