Name:

Math 528: Exam 2

1. (20 pts) Let n be a positive integer. Suppose V, W are vector spaces over F with $\dim V = n$, $\dim W = n+1$, and $T : V \to W$, $S : W \to V$ are linear transformations. Prove that TS is not an isomorphism.

Solution: As $\operatorname{Im} S \subseteq V$, $\dim \operatorname{Im} S \leq n$. By the Rank-Nullity Theorem,

$$\begin{aligned} \operatorname{null}(S) &= \dim W - \operatorname{rk} S \\ &= (n+1) - \dim \operatorname{Im} S \\ &\geq (n+1) - n \\ &= 1. \end{aligned}$$

In particular, S is singular. Let v be a nonzero element of $\ker(S)$. Then

$$(TS)(v) = T(S(v)) = T(0) = 0.$$

Therefore v is a nonzero element of $\ker(TS)$, so TS is not injective, and hence not an isomorphism.

Note that a few people tried to argue that "if TS is an isomorphism, then T and S are isomorphisms." In general this is false! For instance, take $T : \mathbb{R}^3 \to R^2$ given by $T(x, y, z) = (x, y)$ and $S : \mathbb{R}^2 \to \mathbb{R}^3$ given by $S(x, y) = (x, y, 0)$. Then TS is the identity on $\mathbb{R}^2$.

2. (20 pts) Suppose m, n are positive integers, and V, W are vector spaces over F with $\dim V = m$ and $\dim W = n$. Prove that $V \cong W$ if and only if $m = n$.

Solution: See the homework solution.

3. (15 pts) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map satisfying $T(1, 2) = (2, 1)$ and $T(1, -1) = (-1, 0)$. Find the matrix of T with respect to the standard basis.

Solution: Let $\mathscr{B}$ be the standard basis and $\mathscr{B}' = ((1, 2), (1, -1))$. Observe that $(1, 2)$ and $(1, -1)$ are not parallel, so $\mathscr{B}'$ is a basis for $\mathbb{R}^2$. We have

$$(2, 1) = (1, 2) + (1, -1) \text{ and } (-1, 0) = -\frac{1}{3}(1, 2) - \frac{2}{3}(1, -1).$$

Therefore

$$[T]_{\mathscr{B}'} = \begin{bmatrix} 1 & -\frac{1}{3} \\ 1 & -\frac{2}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & -1 \\ 3 & -2 \end{bmatrix}.$$

The change of basis matrix from $\mathscr{B}'$ to $\mathscr{B}$ is

$$P = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

which has inverse

$$P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

We have

$$\begin{aligned}
[T]_{\mathscr{B}} &= P[T]_{\mathscr{B}'}P^{-1} \\
&= \frac{1}{9}\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}\begin{bmatrix}3 & -1\\ 3 & -2\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix} \\
&= \frac{1}{9}\begin{bmatrix}0 & 9\\ 3 & 3\end{bmatrix} \\
&= \begin{bmatrix}0 & 1\\ \frac{1}{3} & \frac{1}{3}\end{bmatrix}.
\end{aligned}$$

4. (15 pts) Let $I \subseteq F[x]$ be

$$\{f \in F[x] : f(1) = 0\}.$$

Prove that I is an ideal, and find a generator.

Solution: One can show that I is an ideal directly, but it is easier to show that $I = (x-1)$; we already know that $(x-1)$ is an ideal, so the equality means I is an ideal. The equality follows from the Root-Factor theorem: $f \in I$ if and only if 1 is a root of f if and only if $(x-1) \mid f$ if and only if $f \in (x)$.

5. (15 pts) Let $L : \mathbb{R}^\infty \to \mathbb{R}^\infty$ be the left shift operator; that is,

$$L(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

Find all eigenvalues of L.

Solution: Every real number is an eigenvalue! For take $\lambda \in \mathbb{R}$. Let

$$x = (1, \lambda, \lambda^2, \lambda^3, \ldots).$$

Note that since the first coordinate of x is 1, $x \neq 0$. But

$$\begin{aligned}
Lx &= (\lambda, \lambda^2, \lambda^3, \lambda^4, \ldots) \\
&= \lambda(1, \lambda, \lambda^2, \lambda^3, \ldots) \\
&= \lambda x.
\end{aligned}$$

This proves the claim.