

Name:

**Math 528: Exam 1**

1. (15 pts) Find all solutions in  $\mathbb{C}$  to the system of equations

$$(1 - i)x - iy = 0$$

$$2x + (1 - i)y = 0.$$

**Solution:** See homework solutions.

2. (15 pts) Prove that if  $U, W$  are subspaces of a vector space  $V$  and  $U \cup W$  is a subspace, then  $U \subseteq W$  or  $W \subseteq U$ .

**Solution:** Suppose for the sake of contradiction  $U \not\subseteq W$  and  $W \not\subseteq U$ . Then  $\exists u \in U$  with  $u \notin W$  and  $w \in W$  with  $w \notin U$ . We have  $u, w \in U \cup W$ , so  $u + w \in U \cup W$  since the latter is a subspace. Thus  $u + w \in U$  or  $u + w \in W$ . Without loss of generality, say  $u + w \in U$ . Then since  $u \in U$ ,

$$(u + w) - u = w$$

also lies in  $U$ . But this contradicts our assumption that  $w \notin U$ . Therefore our assumption was false, and so the claim is true.

3. (15 pts) Let  $A$  and  $B$  be the matrices given below. You are given that  $A$  and  $B$  are row-equivalent. Find an invertible matrix  $P$  for which  $PA = B$ .

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ -1 & 1 & -3 & 8 \\ 0 & -1 & 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

**Solution:** We find a sequence of row operations that turns  $A$  into  $B$ . To do this, add row 1 to row 2, multiply row 2 by  $\frac{1}{3}$ , then add row 2 to row 3. The elementary matrices that accomplish these operations are, respectively,

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Therefore  $E_3 E_2 E_1 A = B$ . Setting  $P$  to the product of the  $E_i$  yields

$$P = E_3 E_2 E_1 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 1 \end{bmatrix}.$$

Note that instead of directly computing the product, we could have also applied those three row operations to the identity matrix to obtain  $P$ .

4. Let  $V$  be the vector space of  $2 \times 2$  matrices over  $\mathbb{R}$ . Let  $W_1$  be the set of matrices of the form  $\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$  and  $W_2$  the set of matrices of the form  $\begin{bmatrix} a & b \\ -a & c \end{bmatrix}$ . Assume that  $W_1$  and  $W_2$  are subspaces of  $V$ . Below, make sure to prove your answers!

(a) (15 pts) Compute  $\dim W_1$ .

**Solution:** It is 3. A basis is given by

$$A_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

To show that this is true, we first show spanning:

$$xA_1 + yA_2 + zA_3 = \begin{bmatrix} x & -x \\ y & z \end{bmatrix},$$

and hence for any matrix in  $W_1$ , we can write it in terms of the  $A_i$ . Next we show linear independence. If  $xA_1 + yA_2 + zA_3 = 0$ , then from the above calculation,

$$\begin{bmatrix} x & -x \\ y & z \end{bmatrix} = 0,$$

and hence  $x = y = z = 0$ . Linear independence follows. As there are 3 elements in our basis, the dimension calculation is correct.

(b) (15 pts) Assuming that  $\dim W_2 = 3$ , compute  $\dim(W_1 + W_2)$ .

**Solution:** We need to compute  $\dim W_1 \cap W_2$ . Let  $A \in W_1 \cap W_2$ . Then for some  $a, b, c, x, y, z \in F$ , we have

$$A = \begin{bmatrix} x & -x \\ y & z \end{bmatrix} = \begin{bmatrix} a & b \\ -a & c \end{bmatrix}.$$

Hence  $a = x$ ,  $b = -x$ ,  $y = -a = -x$ , and  $z = c$ . I claim that

$$B_1 = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \text{ and } A_3$$

form a basis for  $W_1 \cap W_2$ . The proof is similar to part (a); I omit the details here. In any case,  $\dim W_1 \cap W_2 = 2$ . Therefore

$$\begin{aligned} \dim(W_1 + W_2) &= \dim W_1 + \dim W_2 - \dim W_1 \cap W_2 \\ &= 3 + 3 - 2 \\ &= 4. \end{aligned}$$

5. (15 pts) Prove that if  $A$  is a  $3 \times 1$  matrix and  $B$  is a  $1 \times 3$  matrix, then  $AB$  is not invertible.

**Solution:** I take a similar approach as in my homework problem. Let  $B = [b_1, b_2, b_3]$ . If  $b_1 = 0$ , then

$$B \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0,$$

and hence

$$AB \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0.$$

If  $AB$  is invertible, then multiplying both sides by  $(AB)^{-1}$  yields a contradiction.

Now suppose  $b_1 \neq 0$ . Then

$$B \begin{bmatrix} -b_2 \\ b_1 \\ 0 \end{bmatrix} = 0,$$

and a similar proof shows that  $AB$  is not invertible in this case as well.