Exam 1

You may not use notes or other outside resources for this exam. You may not use the result of any homework problem. You may use any result covered in lecture or in the relevant sections of the text.

Throughout, $F$ is a field and $V$ is a vector space over $F$.

1. (a) If $A \in M_{m \times n}(F), B \in M_{n \times m}(F)$, and $m > n$, show that $AB$ is not invertible.

Solution: One can directly show that any row of $AB$ is in the row space of $B$, and hence $\text{rk}(AB) \leq n < m$. But a better way is to use the rank-nullity theorem. The rank of $B$ is at most $n$, and so by the rank-nullity theorem, $\text{null}(B) = m - n > 0$. Let $v$ be a nonzero vector in the nullspace of $B$. Then

\begin{align*}
(AB)(v) &= A(Bv) \\
&= A \cdot 0 \\
&= 0.
\end{align*}

Therefore $v$ is a nonzero vector in the nullspace of $AB$. If $AB$ were invertible, we could multiply both sides of $(AB)v = 0$ by the inverse to obtain $v = 0$, yielding a contradiction. The conclusion follows.

(b) Show by example that if $m < n$, the product may be invertible.

Solution: Over $\mathbb{R}$, let

$$A = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$ 


2. Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}.$$ 

(a) Find a basis for the row space of $A$. 

Solution: The rref of $A$ is
\[
\begin{bmatrix}
1 & 0 & -5 \\
0 & 1 & 3 \\
0 & 0 & 0
\end{bmatrix}.
\]
Thus $(1,0,-5)$ and $(0,1,3)$ form a basis for the row space.

(b) Determine if $(1,3,2)$ lies in the row space.

Solution: If $a(1,0,-5) + b(0,1,3) = (1,3,2)$, we must have $a = 1$ and $b = 3$. But with these choices, the equality does not hold, and so $(1,3,2)$ does not lie in the row space.

3. Let $V = \mathbb{R}^2$. Define an operation of $\mathbb{C}$ on $V$ by
\[
(a + bi)(v) = av + \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} v,
\]
where $a, b \in \mathbb{R}$ and $v \in V$. Show that $V$ is a vector space over $\mathbb{C}$, with scalar multiplication given by the above operation.

Solution: A slick way of doing this is by defining a map $f : V \to \mathbb{C}$ by $f(x,y) = x + iy$, and then observing that for $u, v \in V, z \in \mathbb{C}$, we have
\[
f(u + v) = f(u) + f(v) \text{ and } f(zv) = zf(v).
\]
But I will do it directly. We already know that $V$ is a vector space over $\mathbb{R}$, and hence the additive properties of a vector space are already known. Also, we have $1 \cdot v = v$ for all $v \in V$. Therefore there are only three conditions we need to check: $(wz)v = w(zv), z(u + v) = zu + zv,$ and $(w + z)v = wv + zv$. Let $z = a + bi$ and $w = c + di$. Let
\[
J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\]
It is easy to check that $J^2 = -I$.

For associativity, we have
\[
(wz)v = ((c + di)(a + bi))v \\
= ((ac - bd) + (ad + bc)i)v \\
= (ac - bd)v + (ad + bc)Jv.
\]
On the other hand,
\[
w(zv) = (c + di)(av + bJv) \\
= cav + dj(av) + cbJv + dj(bJv) \\
= cav + adJv + bcJv + bdJ^2v \\
= acv - bdv + adJv + bcJv \\
= (ac - bd)v + (ad + bc)Jv.
\]
Here, we are repeatedly using linearity of the multiplication \(Jv\). We have thus obtained \((wz)v = w(zv)\).
The distributivity properties are easier than the above calculation, so I omit them.

4. Let \(F = \mathbb{R}\) and let \(V\) be the set of polynomials of degree \(\leq 2\).

(a) Let \(t \in \mathbb{R}\). Show that \((1, x + t, (x + t)^2)\) is a basis for \(V\).

**Solution:** For \(a, b, c \in \mathbb{R}\), suppose \(0 = a + b(x + t) + c(x + t)^2\). The coefficient of \(x^2\) on the right-hand side is \(c\), so \(c = 0\). Plugging in \(x = -t\), we get \(a = 0\). But then we must have \(b = 0\). Thus the vectors are linearly independent. Since \((1, x, x^2)\) spans \(V\), we have \(\dim V \leq 3\). But we have a linearly independent set of size 3, so \(\dim V \geq 3\), and hence we have equality. We also conclude that the given list of vectors, since it has 3 elements, is a basis.

(b) Let \(p(x) = a + bx + cx^2\). Compute the coordinates for \(p\) in terms of the above basis.

**Solution:** It suffices to find the change of basis matrix with respect to \((1, x, x^2)\). Let \(B = (1, x, x^2)\) and \(B' = (1, x + t, (x + t)^2)\). We can find a matrix for \(B'\) in terms of \(B\) and invert it, but I will instead find a matrix for \(B\) in terms of \(B'\). Observe that
\[
1 = 1 \cdot 1 \\
x = (-t) \cdot 1 + 1 \cdot (x + t) \\
x^2 = t^2 \cdot 1 + (-2t) \cdot (x + t) + 1 \cdot (x + t)^2.
\]
Let \(Q\) be the matrix
\[
\begin{bmatrix}
1 & -t & t^2 \\
0 & 1 & -2t \\
0 & 0 & 1
\end{bmatrix}.
\]
From our change-of-basis theorem, we get $c_{B'} = Qc_B$. Since $c_B(p) = (a, b, c)$, our answer is

$$c_{B'}(p) = Q\begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$ 

5. Let $V = M_n(F)$ and fix $B \in V$. Define

$$T : V \to V$$

$$T(A) = AB - BA.$$

(a) Show that $T$ is a linear transformation.

**Solution:** For $A_1, A_2 \in V$,

$$T(A_1 + A_2) = (A_1 + A_2)B - B(A_1 + A_2)$$

$$= A_1B + A_2B - BA_1 - BA_2$$

$$= A_1B - BA_1 + A_2B - BA_2$$

$$= T(A_1) + T(A_2).$$

Scalar multiplication is even easier and is omitted.

(b) What choices of $B$ result in $\ker(T) = \{0\}$?

**Solution:** For no choice! We have $T(I) = IB - BI = B - B = 0$, so the identity matrix always lies in the kernel.