HW 8

Prof. Shahed Sharif

§3.1/1, 2a–c, 5, 7a; §3.2/4

3.1.1 For (a), I is a group under addition, and $k[x_{\ell+1}, \ldots, x_n] \subset k[x_1, \ldots, x_n]$ is a subgroup with respect to addition. As the intersection of subgroups is a subgroup, I_ℓ is an additive subgroup. Next, let $f \in I_\ell$ and $r \in k[x_{\ell+1}, \ldots, x_n]$. Then $f \in I$, so $rf \in I$. But $r, f \in k[x_{\ell+1}, \ldots, x_n]$, which is closed under multiplication. Thus $rf \in k[x_{\ell+1}, \ldots, x_n]$. Hence $rf \in I_\ell$. The conclusion follows.

Part (b) follows from the associativity of intersections.

3.1.2 A routine Gröbner basis calculation in Sage yields $I \cap k[x] = \langle x^4 - 4x^2 + 3 \rangle$, $I \cap k[y] = \langle y^3 - y \rangle$. For (b), from $I \cap k[x]$ we have $x = \pm\sqrt{3}, \pm 1$. However, y depends on x; plugging each value into the original pair of equations, we obtain the 4 points

$$(\pm\sqrt{3}, 0), (1, 1), (-1, -1).$$

The last two are in $\mathbb{Q}^2$.

3.2.4 The Gröbner basis is

$$x^2 + 3, y^2 + z^2 - 1.$$

Thus $V(I_1) = V(y^2 + z^2 - 1)$ is the unit circle in the yz-plane. But $V(I)$ is two copies of the circle: one each for $x = \pm\sqrt{-3}$. Claim (a) follows.

Over $\mathbb{R}$, since $\pm\sqrt{-3} \notin \mathbb{R}$, $V(I) = \varnothing$. But $V(I_1)$ are the (real) points on the unit circle with equation $y^2 + z^2 = 1$, which is infinite.