HW 6
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HW 6: §2.4/1, 7, 9; §2.5/1, 3a, 12, 13

2.4.9 I only cover well-ordered. It suffices to show that $(\alpha, \beta) \geq 0$ for all $(\alpha, \beta) \in \mathbb{Z}^n_{\geq 0} \times \mathbb{Z}^m_{\geq 0}$. If $\alpha >_{lex} 0$, then certainly $(\alpha, \beta) > 0$. If not, since lex is a monomial ordering, by Corollary 6 $\alpha = 0$. By the same Corollary, $\beta \geq_{grlex} 0$, and it follows that $(\alpha, \beta) \geq 0$.

2.5.3 Choose $m \in \langle LT(I) \rangle \setminus \langle LT(f_1), \ldots, LT(f_s) \rangle$. By definition of $\langle LT(I) \rangle$, $\exists g \in I$ such that $LT(g) = m$. Let f be the remainder of g upon division by the f_i. As $LT(f_i) \nmid LT(g) = m$ for all i, we have that $f \neq 0$, since from the division algorithm $LT(f) = m$. The claim follows.

2.5.12 We will prove the contrapositive. Suppose there is an ideal I which is *not* finitely generated. Choose $f_1 \in I$. Inductively choose f_i as any element in $I \setminus \langle f_1, \ldots, f_{i-1} \rangle$; such an element must exist, since on the one hand $f_j \in I$ for $1 \leq j \leq i-1$ so $\langle f_1, \ldots, f_{i-1} \rangle \subset I$, but on the other hand we cannot have equality else I would be finitely generated. It follows that the $\langle f_1, \ldots, f_i \rangle$ form an ascending chain which does not stabilize.