# HW 5

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Homework due 10/4: §2.2/4, 10, 11, 12; §2.3/3 (for us, "check" → "compute"), 5, 9

2.2.4 For total ordering, the total degree is the usual ordering on $\mathbb{N} \cup \{0\}$, and since ties are broken by lex, which we know to be total, grlex is also a total ordering.

For additivity, note that $|\alpha + \gamma| = |\alpha| + |\gamma|$ for all exponent vectors $\alpha, \gamma$. Thus if $|\alpha| > |\beta|$, then $|\alpha + \gamma| > |\beta + \gamma|$. In the case of ties, the total degree is still tied after adding $\gamma$. Additivity in this case follows from additivity of lex.

Finally we show grlex is a Well-Ordering. Let $S$ be a nonempty set of exponent vectors. The set of total degrees of $S$ is a subset of $\mathbb{N} \cup \{0\}$, which is Well-Ordered, and hence there is a lowest total degree $d$ amongst elements of $S$. Let $S_d \subset S$ be the set of exponent vectors with total degree $d$; it is nonempty by definition of $d$. Within $S_d$, grlex is equivalent to lex, which is a Well-Ordering. Hence there is a least element $\alpha$ of $S_d$. Note that for $\beta \in S - S_d$, $\alpha < \beta$ as well since $|\beta| > d = |\alpha|$, so $\alpha$ is in fact the least element of $S$.

2.2.11 Let $f = \sum_{\alpha \in A} c_\alpha x^\alpha$, and let $\alpha_0 \in A$ be the largest element of $A$, so that $\alpha < \alpha_0$ for $\alpha \in A - \{\alpha_0\}$. For (a), let $m = x^\gamma$. Then

$$mf = \sum_{\alpha \in A} c_\alpha x^{\alpha + \gamma}.$$

As $\alpha_0 + \gamma > \alpha + \gamma$ for $\alpha \in A - \{\alpha_0\}$, the leading term is $c_{\alpha_0} x^{\alpha_0 + \gamma}$, from which the claim follows.

For (b), this is true. Let $g = \sum_{\beta \in B} d_\beta x^\beta$ with leading term $d_{\beta_0} x^{\beta_0}$. Then

$$fg = \sum_{\substack{\alpha \in A \\ \beta \in B}} c_\alpha d_\beta x^{\alpha + \beta}.$$

For $\alpha \neq \alpha_0$, $\beta \neq \beta_0$,

$$\begin{aligned} \alpha_0 + \beta_0 &> \alpha + \beta_0 \\ &> \alpha + \beta \end{aligned}$$

via additivity. Similarly, $\alpha_0 + \beta_0 > \alpha_0 + \beta$. Therefore the leading term of $fg$ is $c_{\alpha_0} d_{\beta_0} x^{\alpha_0 + \beta_0}$, from which the claim follows.

For (c), this is false: under the usual degree, take $s = 2$, $f_1 = x + 1$, $f_2 = x$, $g_1 = 1$, $g_2 = -1$.

2.3.3 The answers, grlex then lex, are:

$$1a : x^7 + x^3 - y + 1, 2y^3 - y + 1$$
$$1b : x^7 + x^3 - y + 1, y^{23} + y^{11} - y + 1$$
$$2a : x^2 + xy - yz, z \text{ for all 3 permutations}$$