HW 3 Selected solutions

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Homework due 9/20: 1.4/15, 16; 1.5/2, 6, 11; plus SAGE problems.

1.4.15 For (a), the proof is identical to that shown in class when S is a variety.

For (b), I claim that $I(X) = \langle x - y \rangle$. Certainly $\langle x - y \rangle \subset I(X)$. Now suppose $f \in I(X)$. I first claim that $\exists g \in k[x, y], h \in k[y]$ such that

$$f = g \cdot (x - y) + h.$$

I will show this by induction on the the highest exponent of x that appears in the monomial expansion of f; call this exponent n. Certainly if $n = 0$, we can set $g = 0$ and $h = f$. Now suppose the claim holds for the $n - 1$ case. We write

$$f = c_0 x^n + c_1 x^{n-1} + \cdots + c_n$$

where $c_i \in k[y]$ for $0 \le i \le n$. By inductive hypothesis,

$$c_1 x^{n-1} + \cdots + c_n = (x - y) g_1 + h_1$$

for some $g_1 \in k[x, y]$, $h_1 \in k[y]$.

We have

$$\begin{aligned} x^n &= ((x - y) + y)^n \\ &= (x - y)^n + \binom{n}{1} (x - y)^{n-1} y + \cdots + y^n \\ &= (x - y) g_2 + y^n \end{aligned}$$

where $g_2 = (x - y)^{n-1} + \binom{n}{1} (x - y)^{n-2} y + \cdots + y^{n-1}$. It follows that

$$f = (x - y)(g_1 + g_2) + (h_1 + y^n).$$

The induction claim follows.

Now, suppose $f \in I(X)$. Then $f(a, a) = 0$ for $a \neq 1$. The for each such a,

$$f(a, a) = 0 + h(a).$$

The polynomial h therefore has infinitely many roots, and so $h = 0$. It follows that $(x - y) \mid f$, so $f \in \langle x - y \rangle$. The claim now follows

2.6 By our Corollary from class, the equality $\langle h \rangle = \langle f_2, \ldots, f_s \rangle$ implies that $h \in \langle f_2, \ldots, f_s \rangle$ and $f_i \in \langle h \rangle$ for $2 \le i \le s$. From the same Corollary (but the reverse implication), $\langle f_2, \ldots, f_s \rangle \subset \langle f_1, \ldots, f_s \rangle$.

Now clearly $f_1 \in \langle f_1, \ldots, f_s \rangle$. By the above, $h \in \langle f_1, \ldots, f_s \rangle$. Therefore by the same Corollary,

$$\langle f, h \rangle \subset \langle f_1, \ldots, f_s \rangle.$$

Certainly $f_1 \in \langle f_1, h \rangle$. The usual Corollary shows that $\langle h \rangle \subset \langle f_1, h \rangle$, and so bye the first paragraph above, $f_i \in \langle f_1, h \rangle$ for $2 \leq i \leq s$. It follows that

$$\langle f_1, \ldots, f_s \rangle \subset \langle f_1, h \rangle \, .$$

The claim follows.