HW 2 Selected Solutions

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1.3.6 For (a), the line ℓ through $(0,0,1)$ and $(u,v,0)$ intersects the sphere at one other place, and so yields a map $\rho : \mathbb{R}^2 \to S^2$. It is injective since if $\rho(u_1,v_1) = \rho(u_2,v_2)$, then the corresponding lines ℓ_1, ℓ_2 must intersect S^2 at the same point Q. But there is a unique line through N and Q, and hence $\ell_1 = \ell_2$. This line intersects the xy-plane at a unique point, and hence $(u_1,v_1) = (u_2,v_2)$. For surjectivity away from N, if $Q \in S^2 - \{N\}$, the line through N and Q should intersect the plane the plane at some point (say $(u,v,0)$), and hence $\rho(u,v) = Q$.

If we want $\rho(u,v) = N$, then the line must be tangent to the sphere at N, and hence lies in the tangent plane at N. The tangent plane is given by

$$\nabla(x^2 + y^2 + z^2)_{(0,0,1)} \cdot (x - 0, y - 0, z - 1) = 0,$$

where ∇ denotes the gradient (in this case $(2x, 2y, 2z)$). Simplifying, the tangent plane is $z = 1$. This is parallel to the xy-plane, and hence our line through $(0,0,1)$ cannot be obtained from any $(u,v,0)$.

For (b), this is a 1-parameter linear function (of t), and hence gives a line. Setting $t = 0$ and $t = 1$, we obtain $(0,0,1)$ and $(u,v,0)$, respectively. As the line through these points is unique, it must be the one given by this parametrization.

Part (c) is straightforward algebra.

1.4.1 For (a), first observe that if $x = 0$, then the second equation can never be satisfied. Next, if we temporarily work over the fraction field of $k[x,y]$, we get $y = \frac{1}{x}$ from the second equation. Substituting inoto the first, we obtain $x^2 + \frac{1}{x^2} - 1 = 0$, or $x^4 - x^2 + 1 = 0$. Thus over $\mathbb{C}$, there are 4 points on this variety.

For (b), note first that we must work in $k[x,y]$. Taking the hint, and with some experimentation, we arrive at

$$x^2(x^2 + y^2 - 1) - (xy + 1)(xy - 1) = x^4 - x^2 + 1.$$

1.4.8 For (a), suppose $f^m \in I(V)$. By definition, this means that $f^m(a) = f(a)^m = 0$ for all $a \in V$. But then $f(a) = 0\ \forall a \in V$, and hence $f \in I(V)$ too. The claim follows.

For (b), let $f = x$. Then clearly $f^2 \in \langle x^2, y^2 \rangle$. We show that $f \notin \langle x^2, y^2 \rangle$. For if it was, we'd have

$$x = h_1 x^2 + h_2 y^2$$

for some $h_1, h_2 \in k[x,y]$. In particular, this equality must hold if we substitute $y = 0$; thus, $x = h_1(x,0)x^2$. But this equality takes place in $k[x]$, and as the degree on the right side is strictly greater than that on the left, we obtain a contradiction. Therefore $f \notin \langle x^2, y^2 \rangle$. It follows that $\langle x^2, y^2 \rangle$ is not radical.