HW 1 Selected Solutions

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6 For a, take $V(x_1 - a_1, \dots, x_n - a_n)$. For b, a finite set is a finite union of single points. By a, each single point is a variety, and by 15a, finite unions of varieties are varieties. The claim follows.

7 Let $P = (x_0, y_0) = (r_0; \theta_0)$. For a, suppose P is on the four-leaved rose, so that $r_0 = \sin 2\theta_0 = 2\sin\theta_0\cos\theta_0$. Multiplying through by r_0^2, we obtain

$$r_0^3 = 2r_0 \sin\theta_0 r_0 \cos\theta_0.$$

Squaring both sides, we get $r_0^6 = 4(r_0 \sin\theta_0)^2(r_0\cos\theta_0)^2$, which implies that $(x_0^2 + y_0^2)^3 = 4x_0^2y_0^2$. This is precisely the equation for the given variety, and so P lies on that variety.

For b, say P is on the affine variety, so that $(x_0^2 + y_0^2)^3 = 4x_0^2y_0^2$. If P is the origin, then certainly P lies on four-leaved rose. Therefore assume otherwise; in particular, $r_0 = \sqrt{x_0^2 + y_0^2} \neq 0$. Then we get $r_0^6 = 4(r_0\sin\theta_0)^2(r_0\cos\theta_0)^2$. As $r_0 \neq 0$, we can divide out to get

$$r_0^2 = 4\sin^2\theta_0\cos^2\theta_0.$$

Thus we have $r_0 = \pm 2\sin\theta_0\cos\theta_0 = \pm\sin 2\theta_0$. But notice that the four-leaved rose has 180° symmetry (for instance, the map $\theta \mapsto \theta + \pi$ leaves the r unchanged). Therefore if $(r_0; \theta_0)$ satisfies either equation, P is on the four-leaved rose.

15a We showed the case of 2 varieties. Let us consider the inductive step for unions. Suppose the union of $n - 1$ varieties is a variety. Let $V_1, \dots, V_n$ be varieties. We have

$$V_1 \cup V_2 \cup \cdots \cup V_n = (V_1 \cup V_2 \cup \cdots \cup V_{n-1}) \cup V_n.$$

By inductive hypothesis, the parenthetical term is a variety; call it W. By the $n = 2$ case, $W \cup V_n$ is a variety. The claim follows.

The argument for intersections is similar.

15b For $a \in \mathbb{R}$, let $V_a = V(x - a) = \{a\} \subset \mathbb{R}$. I claim that $W = \cup_{a\neq 0} V_a = \mathbb{R} - \{0\}$. This is an infinite union of varieties. Suppose that W is a variety; say $W = V(f_1, \dots, f_s)$ where $f_1, \dots, f_s \in \mathbb{R}[x]$. For each i, we get that $f_i(a) = 0$ for all $a \in W$, and hence f_i has infinitely many roots. This implies that f_i is the zero polynomial. Thus $W = V(0)$. But $V(0) = \mathbb{R} \neq W$, so we have a contradiction. It follows that W is not an affine variety.