Name:

Math 521: Exam 1b

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. Calculators are not allowed.

You may use any result from the chapters covered in the text or from lecture. You may not use the results of homework or worksheet problems.

1. (10 pts) Under the grlex ordering on $k[x,y,z]$, divide $yz^2+y^2-z^2$ by $[xz-y^2, yz+x]$.

Solution: Routine; the quotient are -1 and z, and the remainder is $-z^2$.

2. Let I,J be ideals in $k[x_1,\ldots,x_n]$. Define IJ to be the ideal generated by $\{f\cdot g : f\in I, g\in J\}$.

   (a) (10 pts) Let $I=\langle f_1,\ldots,f_s\rangle$ and $J=\langle g_1,\ldots,g_t\rangle$. Show that $IJ=\langle\{f_ig_j\}_{\substack{i=1,\ldots,s\\ j=1,\ldots,t}}\rangle$

Solution: Certainly $f_ig_j\in IJ$, so $\langle\{f_ig_j\}\rangle\subset IJ$. Now let $fg\in IJ$; specifically, let $f\in I$ and $g\in J$. Then $\exists a_1,\ldots,a_s,\ b_1,\ldots,b_t\in k[x_1,\ldots,x_n]$ such that

$$f=\sum a_if_i \text{ and } g=\sum b_ig_i.$$

Thus

$$fg=\sum_{i,j}a_ib_jf_ig_j.$$

It follows that $fg\in\langle\{f_ig_j\}_{\substack{i=1,\ldots,s\\ j=1,\ldots,t}}\rangle$. We conclude the desired equality.

   (b) (5 pts) Show that $IJ\subset I\cap J$.

Solution: It suffices to show that $f_ig_j\in I\cap J$ for all i,j. As $f_i\in I$, by absorption $f_ig_j\in I$. Similarly, $f_ig_j\in J$. The claim follows.

   (c) (10 pts) Give an example to show that IJ need not equal $I\cap J$. Show that your example works.

Solution: Let $n=1$, $I=J=\langle x\rangle$. Then $IJ=\langle x^2\rangle$ while $I\cap J=\langle x\rangle$. As $x^2\nmid x$, $x\notin IJ$. Thus $I\cap J\not\subset IJ$.
There are of course many other examples.

   (d) (15 pts) Prove that $V(IJ)=V(I)\cup V(J)$.

Solution: As $IJ\subset I\cap J$, we have $IJ\subset I$, and hence $V(I)\subset V(IJ)$. Similarly, $V(J)\subset V(IJ)$. Therefore $V(I)\cup V(J)\subset V(IJ)$.
For the other direction, suppose $P\in V(IJ)$. Let $f_1,\ldots,f_s,\ g_1,\ldots,g_t$ be as above (we can assume this is the case by Hilbert Basis Theorem, but this justification was not necessary). If $g_j(P)=0$ for all j, then $P\in V(J)$, and we are done. Otherwise, $g_j(P)\neq 0$ for some j; without loss of generality for $j=1$. Consider the products f_ig_1: we have $(f_ig_1)(P)=0$ since $f_ig_1\in IJ$, but $g_1(P)\neq 0$, and hence $f_i(P)=0$ for all i. Thus $P\in V(I)$. It follows that $V(IJ)\subset V(I)\cup V(J)$, and we are done.

Actually, you did not need to do this all out; we showed that $V(I) \cup V(J) = V(\{f_i g_j\})$ in class, so you can just cite that result.

3. (10 pts) Show that on $k[x, y]$, grlex and grevlex are the same; that is $\alpha > \beta$ for grlex if and only if $\alpha > \beta$ for grevlex.

Solution: Let $\alpha = (a_1, a_2)$ and $\beta = (b_1, b_2)$. Of course if $a_1 + a_2 > b_1 + b_2$, then $\alpha > \beta$ in both orderings. Suppose that $a_1 + a_2 = b_1 + b_2$. Suppose $\alpha >_{grlex} \beta$. Then we must have $a_1 > b_1$. But then $a_2 < b_2$, and so $\alpha >_{grevlex} \beta$. The other direction is the same: if $\alpha >_{grevlex} \beta$, then we must have $a_2 < b_2$, so $a_1 > b_1$, and hence $\alpha >_{grlex} \beta$.

4. (10 pts) Let G be a finite set of monomials, and let $I = \langle G \rangle$. Show that G is a Gröbner basis for I.

Solution: First, observe that $LT(m) = m$ for all $m \in G$; hence $\langle G \rangle = \langle LT(G) \rangle$.
Let $f \in I$, $f \neq 0$. Then as I is a monomial ideal, every term of f is divisible by some element of G. In particular, $m \mid LT(f)$ for some $m \in G$. It follows that $\langle LT(I) \rangle \subset \langle LT(G) \rangle$. The reverse inclusion is obvious, so the claim follows.

5. Let $\varphi : \mathbb{C} \to \mathbb{C}^2$ be given by
$$\varphi(t) = (t^2 + t, t).$$
Let W denote the image.
   (a) (5 pts) Compute $I(W)$.

   Solution: $\langle x - y^2 - y \rangle$

   (b) (10 pts) Prove the previous part.

   Solution: Let J be the ideal from part (a). If we plug in $(t^2 + t, t)$ for (x, y) in $x - y^2 - y$, we get 0. Hence $J \subset I(W)$.
   Let $f \in I(W)$. We divide f by $x - y^2 - y$ with respect to lex to obtain
   $$f = q \cdot (x - y^2 - y) + r$$
   with $r \in k[y]$. For any $t \in \mathbb{C}$, when we plug in $P = (t^2 + t, t)$, we get $f(P) = 0$ from $f \in I(W)$, while on the right hand side we get $r(t)$. Thus $r(t) = 0$ for all t, and hence $r = 0$. It follows that $(x - y^2 - y) \mid f$, so $f \in J$.
   The claim follows.

   (c) (10 pts) Show that $W = V(I(W))$.

   Solution: The inclusion $W \subset V(I(W))$ is immediate: for $P \in W$, $f \in I(W)$, $f(P) = 0$.
   Now suppose $P \in V(I(W))$. Write $P = (a, b)$. As $P \in V(J)$, $(x - y^2 - y)(P) = 0$, so $a - b^2 - b = 0$, so $a = b^2 + b$. Therefore $\varphi(b) = (b^2 + b, b) = P$, and hence $P \in W$.
   The claim follows.