Math 520: Exam 1

February 20, 2020

Make sure to show all your work as clearly as possible. This includes proving your answers, unless otherwise stated!

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1. Short answer questions. There is no partial credit for these, and you need not show any work.

(a) (5 points) Give an example of a ring which is a UFD but not a PID.

**Solution:** \( \mathbb{R}[x, y] \)

(b) (5 points) Give an example of a ring \( R \) and a nonzero ideal \( I \subset R \) so that \( I \) is prime but not maximal.

**Solution:** \( R = \mathbb{R}[x, y], I = (x) \).

(c) (5 points) Let \( M, N \) be left \( R \)-modules. Give the definition of an \( R \)-module homomorphism \( M \to N \).

**Solution:** A group homomorphism \( \varphi : M \to N \) such that \( \forall r \in R, m \in M, \varphi(rm) = r\varphi(m) \).

2. (15 points) Let \( R \) be a PID and \( I \subset R \) an ideal. Show that there are only finitely many ideals \( J \) containing \( I \).

**Solution:** Since \( R \) is a PID, \( I = (\alpha) \) for some \( \alpha \in R \). Suppose \( J \supset I \); as with \( I, J = (\beta) \) for some \( \beta \in R \). Then \( \beta \mid \alpha \). Conversely, if \( \beta \mid \alpha \), then \( (\beta) \supset (\alpha) \). If \( \beta' \) is an associate of \( \beta \), then \( \beta' \mid \alpha \), but \( (\beta') = (\beta) \). Thus, it suffices to show that the number of divisors of \( \alpha \) up to associate is finite.

As \( R \) is a PID, it is also a UFD. Thus we may write \( \alpha \) as a product of irreducibles \( p_1 \cdots p_r \).

Suppose \( \beta \) is a “sub-product”; that is, \( \beta = \prod_{i \in S} p_i \), where \( S \subset \{1, 2, \ldots, r\} \). Then certainly \( \beta \mid \alpha \). There are finitely many subsets \( S \), and thus finitely many such \( \beta \). I claim that any divisor of \( \alpha \) must be an associate of one of these \( \beta \).

For suppose \( \gamma \mid \alpha \), so that \( \gamma \cdot \delta = \alpha \) for some \( \delta \). Taking the prime factorizations of \( \gamma \) and \( \delta \), then invoking uniqueness, we see that if

\[
\gamma = q_1 \cdots q_s
\]

with the \( q_i \) irreducible, then \( s \leq r \) and, after reordering, \( q_i \) is associate to \( p_i \). Write \( q_i = u_i p_i \) where \( u_i \) is a unit, and let \( u = \prod u_i \). Then

\[
\gamma = up_1 \cdots p_s.
\]

The claim follows.

3. (15 points) Let \( R, S \) be commutative unital rings, and \( K \subset R \times S \) an ideal. Show that \( K = I \times J \) for some ideals \( I \subset R, J \subset S \).

**Solution:** See homework solution.

4. (15 points) Let \( R \) be a ring, and define a (left) module action of \( R \) on itself by

\[
r \cdot x = rxx^{-1}.
\]

Show that this action is well-defined; that is, it satisfies the axioms for left modules.
Solution: I made a mistake here: the action does not satisfy \((r+s) \cdot x = r \cdot x + s \cdot x\). Thus, this is not a well-defined module action.

5. (15 points) Let \(R = \mathbb{R}[x], M = \mathbb{R}^2\), and make \(M\) into an \(R\)-module by letting constants act by scalar multiplication, and \(x\) act via multiplication on the left by the matrix

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

Show that the annihilator of \(M\) is the ideal \((x^2 - 1)\); that is,

\[(x^2 - 1) = \{ f \in R \mid \forall m \in M, f \cdot m = 0 \}.
\]

Solution: Let \(A\) denote the matrix above. It is easy to check that \(A^2 - I = 0\). Thus for all \(m \in M\), \((x^2 - 1) \cdot m = 0\). Similarly, if \(f(x) = g(x) \cdot (x^2 - 1)\), then

\[f(x) \cdot m = g(x) \cdot ((x^2 - 1) \cdot m) = g(x) \cdot 0 = 0.\]

This shows that \((x^2 - 1)\) is contained in the annihilator.

Now let \(f(x)\) be in the annihilator. By the division algorithm, we can find \(q(x), r(x)\) with \(\deg r(x) \leq 1\) such that

\[f(x) = q(x)(x^2 - 1) + r(x).\]

Observe that for \(m \in M\),

\[0 = f(x) \cdot m = q(x)(x^2 - 1) \cdot m + r(x) \cdot m = r(x) \cdot m.\]

Let \(r(x) = ax + b\). Let \(m = (1, 0)\). Then

\[r(x) \cdot m = ax \cdot (1, 0) + b \cdot (1, 0)
= a \cdot (0, 1) + b \cdot (1, 0)
= (b, a).
\]

Since \(r(x) \cdot m = 0\), we have \(a = b = 0\), and hence \(r(x) = 0\). Therefore \(f(x) \in (x^2 - 1)\). The claim follows.

6. Let \(M, N\) be left \(R\)-modules and \(\varphi : M \to N\) an \(R\)-module homomorphism.

(a) (10 points) Show that \(\ker \varphi\) is an \(R\)-submodule of \(M\).

Solution: Kernels are subgroups, so it suffices to show that \(\ker \varphi\) is closed under the \(R\)-action. Let \(m \in \ker \varphi\), so \(\varphi(m) = 0\). Then for \(r \in R\),

\[
\varphi(rm) = r\varphi(m)
= r \cdot 0
= 0,
\]

and so \(rm \in \ker \varphi\). The claim follows.
(b) (10 points) Show by example that if $\varphi$ is a homomorphism, but not an $R$-module homomorphism, then $\ker \varphi$ need not be an $R$-submodule of $M$.

Solution: Let $R = \mathbb{Z} \times \mathbb{Z}$, $M = R$ considered as a module via left multiplication, and $\varphi : M \to M$ given by $\varphi(a, b) = (a - b, a - b)$. The map $\varphi$ can be encoded as matrix multiplication by

$$
\begin{bmatrix}
1 & -1 \\
1 & -1
\end{bmatrix},
$$

and hence is a homomorphism. The kernel is

$$\{(m, m) \mid m \in \mathbb{Z}\}.$$

In particular, $(1, 1) \in \ker \varphi$. But $(1, 2) \cdot (1, 1) = (1, 2) \notin \ker \varphi$. 