Exam 2

You have an hour to do the following problems. No notes or calculators are allowed. Simplify as much as makes sense to do by hand. Make sure to show your work and/or explain your reasoning.

1. Consider the identity
   \[ \sum k \binom{n}{k} = n2^{n-1}. \]

   (a) Prove the identity algebraically. (Hint: use the binomial theorem.)

   **Solution:** Take the derivative on both sides of the binomial theorem to obtain
   \[ n(1+x)^{n-1} = \sum k \binom{n}{k} x^{k-1}. \]
   Now plug in \( x = 1. \)

   (b) Prove the identity combinatorially. (Hint: think of a committee with chair.)

   **Solution:** We count the number of “chaired committees” of \( n; \) that is, pairs \((x, U)\) where \( U \subseteq n \) is a committee and \( x \in U \) is the chair. If we choose \( x \) first, there are \( n \) ways of choosing \( x \), and then we must choose \( U - \{x\} \). But \( U - \{x\} \) is a subset of \( n - \{x\} \), of which there are \( 2^{n-1} \). This gives the right-hand side of the identity. We now count the number of chaired committees a different way. Let \( S_k \) be the set of \((x, U)\) as above but for which \#\( U = k \). Note that the \( S_k \) for \( 1 \leq k \leq n \) forms a partition of the set of chaired committees. For fixed \( k \), there are \( \binom{n}{k} \) ways of choosing \( U \). There are \( k \) choices of \( x \) from amongst the elements of \( U \). Therefore \#\( S_k = k \binom{n}{k} \). Applying the addition principle now gets us the left-hand side of the equality.

2. Use Newton’s Binomial Theorem to approximate \((101)^{1.5}\). Use a quadratic approximation.
Solution: We have

\[(1 + x)^{1.5} = 1 + \binom{1.5}{1}x + \binom{1.5}{2}x^2 + \cdots = 1 + \frac{1.5}{1}x + \frac{1.5 \cdot 0.5}{2}x^2 + \cdots = 1 + 1.5x + 0.375x^2 + \cdots \approx 1 + 1.5x + 0.375x^2.\]

Furthermore, \(100^{1.5} = 100^{3/2} = \sqrt{100^3} = 1000.\) Thus

\[(101)^{1.5} = 100^{1.5}(1 + .01)^{1.5} \approx 1000(1 + 1.5 \cdot .01 + 0.375 \cdot 0.001) = 1000 + 15 + 0.375 = 1015.0375.\]

3. How many ways are there of placing 8 identical balls in 4 distinct urns so that there are at most 3 balls in urn 1 and at most 3 balls in urn 2?

Solution: Let \(e_i\) be the number of balls in urn \(i\). Then we are looking for integer solutions to

\[e_1 + e_2 + e_3 + e_4 = 8\]

subject to \(0 \leq e_1, e_2 \leq 3\) and \(0 \leq e_3, e_4\). We do this the usual way: compute the number of solutions without the maximum restrictions, then use inclusion-exclusion to remove the “bad” solutions. The total number of solutions with \(e_1\) unbounded above is

\[\binom{8 + 4 - 1}{8} = \binom{11}{8}.\]

The number of bad solutions with \(e_1 \geq 4\) is equal to the number of solutions to \(\sum e_i = 4\), of which there are

\[\binom{4 + 4 - 1}{4} = \binom{7}{4}.\]

The same calculation holds when \(e_2 \geq 4\). Lastly, when both \(e_1\) and \(e_2\) are \(\geq 4\), there is exactly one solution: \((e_1, e_2, e_3, e_4) = (4, 4, 0, 0)\). This yields an answer of

\[\binom{11}{8} - 2 \binom{7}{4} + 1.\]
4. Suppose a sequence \((a_n)\) satisfies 
\[ a_n = a_{n-1} + 2a_{n-2} \text{ for } n \geq 2, \text{ and } a_0 = 2, \ a_1 = 1. \]
Find an explicit formula for \(a_n\).

**Solution:** This is very similar to number 6 on the review worksheet. In fact, only the initial conditions are different. Thus the matrix \(A\) is the same, as are the eigenvalues and eigenvectors. The stage where this problem differs is where we want to solve 
\[ v_0 = au_1 + bu_2 \]
or
\[ \begin{bmatrix} 2 \\ 1 \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \]
You might be able to just eyeball this one and conclude that \(a = b = 1\). Our solution in vector form is therefore 
\[ v_n = (-1)^n u_1 + 2^n u_2. \]
Solving for \(a_n\), we get 
\[ a_n = (-1)^n + 2^n. \]

5. We fill a bag with apples and bananas. The number of apples is even, and there is at least one banana. If \(c_n\) is the number of ways to fill the bag with \(n\) fruits, what is the (ordinary) generating function for \((c_n)\)?

**Solution:** Let \(A(x)\) be the generating function for the number of ways to fill the bag with apples, and \(B(x)\) the generating function for the number of ways to fill the bag with bananas. I leave as an exercise that 
\[ A(x) = 1 + x^2 + x^4 + \cdots \text{ and} \]
\[ B(x) = x + x^2 + x^3 + \cdots. \]
Since \(c_n\) is the convolution of the number of ways to fill the bag with
apples and the number of ways with bananas, we have

$$C(x) = A(x)B(x)$$

$$= (1 + x^2 + x^4 + \cdots)(x + x^2 + x^3 + \cdots)$$

$$= \frac{1}{1-x^2} \cdot \frac{x}{1-x}$$

$$= \frac{x}{(1-x)(1-x^2)}.$$