Worksheet 21

1. Let $R = \mathbb{Z} \times \mathbb{Z}$. Find all elements $x \in R$ which satisfy $x^2 = x$.

2. Define $\mathbb{Z}[D_6]$ to be the set

$$\left\{ \sum_{\alpha \in D_6} c_\alpha \alpha : c_\alpha \in \mathbb{Z} \right\}.$$

   Write 1 for the element $1 \cdot e$.

   (a) Take $x = 2 - \tau$ and $y = \tau + \sigma\tau - 3\sigma^2$. Compute $x + y$ and xy using the natural operations.

   (b) The set $\mathbb{Z}[D_6]$ forms a ring under the two natural operations; you may assume this going forwards. Show that $\mathbb{Z}[D_6]$ is *not* commutative.

   (c) Show that $\mathbb{Z}[D_6]$ is not an integral domain.

3. Define $(\mathbb{Z}/3\mathbb{Z})[D_6]$ to be the set

$$\left\{ \sum_{\alpha \in D_6} c_\alpha \alpha : c_\alpha \in \mathbb{Z}/3\mathbb{Z} \right\}.$$

   Consider it to be a ring in a similar manner as $\mathbb{Z}[D_6]$. Show that $\exists x \in (\mathbb{Z}/3\mathbb{Z})[D_6]$ for which $x \neq 0$, but $x^3 = 0$.