HW 8 Hints and selected solutions

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2.65 Observe that $\gamma_b(ab) = babb^{-1} = ba$, so that ab and ba are conjugate. The result follows from Prop 2.92(ii).

2.69 Show $\forall b \in G$, $bHb^{-1} \subset H$ directly.

2.71 Remember our characterization of conjugation in S_n.

2.82 Just mechanically apply the definitions for all of these.

A. Let G be cyclic of order n; say $G = \langle g \rangle$. Define $\varphi : \mathbb{Z}_n \to G$ by $\varphi(i) = g^i$. This is well-defined, since if $i \equiv j \pmod{n}$, we have $j = i + kn$ for some $k \in \mathbb{Z}$, and

$$\begin{aligned}\varphi(j) &= g^{i+kn} \\ &= g^i g^{kn} \\ &= g^i (g^n)^k \\ &= g^i.\end{aligned}$$

To show that this is a homomorphism, observe that $\varphi(h+i) = g^{h+i} = g^h g^i = \varphi(h)\varphi(i)$. Next, we compute the kernel. Suppose $\varphi(i) = e$. Then $g^i = e$, so $n \mid i$; but then $i \equiv 0 \pmod{n}$. Thus $\ker(\varphi)$ is trivial, and so φ is injective. Both G and $\mathbb{Z}_n$ have cardinality n, and so by the Pigeonhole Principle, φ is surjective.

For the second part, we just need to show that $\mathbb{Z}_{11}^\times$ is cyclic (and has order 10; this part is easy just be listing elements). I claim 2 is a generator. From Lagrange's Theorem, we know that the order is $1, 2, 5$, or 10. $2^2 \equiv 4 \pmod{11}$ while $2^5 = 32 \equiv 10 \pmod{11}$. By process of elimination, the order of 2 is 10, and so $\mathbb{Z}_{11}^\times = \langle 2 \rangle$, and the claim follows.

B. Don't forget the easy subgroups.

C. Show that first group is not cyclic.

D. These groups are small; specifying the isomorphism with a table is a good way to do this.

E. $\mathbb{Z}_{12}, \mathbb{Z}_2 \times \mathbb{Z}_6, D_{12}, A_4$ are pair-wise non-isomorphic, so any 3 will do the trick. To prove it, observe that the first two are abelian, and the last two are not, so neither of the first two can be isomorphic to either of the last two.

The group $\mathbb{Z}_{12}$ has an element of order 12 (namely 1), but for $(a, b) \in \mathbb{Z}_2 \times Z_6$, $6(a, b) = (6a, 6b) = (0, 0)$, so there is no element of order bigger than 6. Thus the first two groups are nonisomorphic.

For the last two, observe that $\sigma \in D_{12}$ has order 6, but in A_4, there are no elements of order 6: the cycle type would have to contain a 6 cycle, or at least one 2 cycle and one 3 cycle. But any element in A_4 can only move up to 4 elements, so neither cycle type can occur.