HW 6 Selected solutions

Prof. Shahed Sharif

2.55 For (i), let $\alpha = (2\,3)$. We have $\langle(1\,2)\rangle = \{(1), (1\,2)\}$, and so

$$\alpha \langle(1\,2)\rangle = \{(2\,3), (1\,3\,2)\}$$

while

$$\langle(1\,2)\rangle \alpha = \{(2\,3), (1\,2\,3)\}.$$

Observe that these are unequal.

For (ii), let G/H be the set of left cosets and $H\backslash G$ the set of right cosets. We will construct a bijection between them; namely, define

$$f : G/H \to H\backslash G$$

by $f(aH) = Ha^{-1}$. I first show that this is well-defined; that is, if $aH = bH$, I need $Ha^{-1} = Hb^{-1}$. From $aH = bH$, we have $a \in bH$, so $\exists h \in H$ such that $a = bh$. Inverting both sides, we obtain $a^{-1} = h^{-1}b^{-1}$. As $h^{-1} \in H$ since H is a subgroup, we see that $h^{-1}b^{-1} \in Hb$, and so $a^{-1} \in Hb$. Since the right cosets form a partition of G, this means that $Ha^{-1} = Hb^{-1}$. Thus f is well-defined.

Finally, we need to show that f is bijective. I claim that the inverse is given by $g(Hb) = b^{-1}H$. The map g is well-defined by a similar argument as for f. We also have $g(f(aH)) = g(Ha^{-1}) = (a^{-1})^{-1}H = aH$, and the other direction is similar. Thus f and g are inverses. Therefore f is a bijection, and so the number of left and right cosets is the same.

There is another way of doing this problem: both left and right cosets form a partition. We know each left coset has the same cardinality as H. It turns out a similar argument shows every right coset has the same cardinality as H (exercise). Then by the proof of Lagrange's Theorem, the number of right cosets is $\#G/\#H$; but this is the same as the number of left cosets.

2.66 We showed (i) in class. For (ii), let $x \in G$ and let $y = f(x)$. From part (i), we know that $\text{order}(y) \mid \text{order}(x)$. From the corollary to Lagrange's Theorem, $\text{order}(x) \mid \#G$, so $\text{order}(y) \mid \#G$. Similarly, $\text{order}(y) \mid \#H$. Thus, $\text{order}(y)$ is a common divisor of $\#G$ and $\#H$. It follows that $\text{order}(y) = 1$, and hence $y = 1$. The claim follows.

C. Let Let $x, y \in G$. Then

$$\begin{aligned}\varphi(x * y) &= \varphi(x + y + 1) \\ &= x + y + 2,\end{aligned}$$

while

$$\begin{aligned}\varphi(x) + \varphi(y) &= (x + 1) + (y + 1) \\ &= x + y + 2.\end{aligned}$$

Therefore it is a homomorphism. (Now that we know isomorphisms, it is a good exercise to show that φ is in fact an isomorphism.)