HW 5 Selected Solutions

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2.38 The condition $x^2 = e$ is equivalent to $x = x^{-1}$. Let $a, b \in G$. Then we have $a^{-1} = a$, $b^{-1} = b$, and $(ab)^{-1} = ab$. But

$$\begin{aligned}(ab)^{-1} &= b^{-1}a^{-1} \\ &= ba.\end{aligned}$$

Therefore $ab = ba$. The claim follows.

A. We have $e(1) = 1$ by definition of the identity, so $e \in H$. Suppose $\alpha \in H$. Then $\alpha(1) = 1$. Applying α^{-1} to both sides, we obtain $1 = \alpha^{-1}(1)$, which shows that $\alpha^{-1} \in H$. Thus H is closed under inversion. Suppose $\alpha, \beta \in H$. Then $\alpha(1) = \beta(1) = 1$. We have

$$\begin{aligned}(\alpha\beta)(1) &= \alpha(\beta(1)) \\ &= \alpha(1) \\ &= 1.\end{aligned}$$

Therefore $\alpha\beta \in H$.

E. The set of rotations is $\langle\sigma\rangle$. Since this is a cyclic subgroup, it is a subgroup! You can prove it directly of course: $\sigma^0 = e \in \langle\sigma\rangle$; for $\sigma^i \in \langle\sigma\rangle$, $(\sigma^i)^{-1} = \sigma^{-i} \in \langle\sigma\rangle$; and $\sigma^i\sigma^j = \sigma^{i+j}$ proves closure.

The set of reflections are those elements of the form $\sigma^i\tau$. This set does not contain the identity, nor is it closed: $(\sigma\tau) \cdot (\tau) = \sigma$, which is a rotation, not a reflection.

F. If $3 \nmid n$, then this set is just the identity. If $3 \mid n$, then this set is $\langle\sigma^{n/3}\rangle = \{e, \sigma^{n/3}, \sigma^{2n/3}\rangle$. To prove this, first note that $\#D_{2n} = 2n$, so if $3 \nmid n$, then $3 \nmid \#D_{2n}$. By the corollary to Lagrange's Theorem, there are no elements of order 3 in D_{2n}. There is always exactly one element of order 1 (and hence period 3); namely the identity.

Now suppose $3 \mid n$. There are two types of elements of D_{2n}: rotations and reflections. Reflections all have order 2, and so they are out. The set of rotations is $\langle\sigma\rangle$. By part C, the elements of order 3 are those of the form σ^k where

$$\frac{n}{\gcd(k, n)} = 3.$$

Thus k must be a multiple of $n/3$, and the conclusion follows.