Math 470: Abstract Algebra Homework 4

Chapter 2 Solutions

Proposition 1 (Exercise 2.19)**.** *Suppose that* $\alpha \in S_9$ *is given by*

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

Then $sgn(\alpha) = 1$ *and* $\alpha^{-1} = \alpha$.

Proof. We observe that

$$\alpha = (19)(28)(37)(46)(5)$$

where the factorization above is complete and all cycles are disjoint. Thus, the cycles commute. By Proposition 2.27,

$$\alpha^{-1} = (91)(82)(73)(64)(5).$$

Noting, also, that transpositions and 1-cycles are their own inverses, we see that $\alpha^{-1} = \alpha$.

Also, $\text{sgn}(\alpha) = (-1)^{9-5} = 1$, by use of the formula from the definition. ■

Proposition 2 (Exercise 2.20)**.** *Suppose that* $\sigma \in S_n$ *fixes some* j*, where* $1 \leq j \leq n$*. Define* $\sigma' \in S_X$ *by* $\sigma'(i) = \sigma(i)$ *for all* $i \neq j$ *where* $X = \{i \in \{1, ..., n\} : i \neq j\}$*. Then* $sgn(\sigma') = sgn(\sigma)$*. (Note that* X *has* $n - 1$ *elements, so, for permutations, it can be seen as essentially having the same properties as* S_{n-1}*.)*

Proof. By Theorems 2.24 and 2.26, σ has a unique complete factorization up to the order of the factors. Say,

$$\sigma = \sigma_1 ... \sigma_r.$$

Since σ fixes j, one of $\sigma_1, ..., \sigma_r$ must be the 1-cycle (j). WLOG, choose $\sigma_r = (j)$. Moreover, j does not appear in any of $\sigma_1, ..., \sigma_{r-1}$. We define the function

$$\sigma'' := \sigma_1 ... \sigma_{r-1}.$$

Note that $\sigma'' : X \to X$. We observe that $\sigma''(i) = \sigma'(i)$ for all $i \in X$. Thus, $\sigma'' = \sigma'$. Hence, σ' and σ'' have the same parity with respect to X. So, we have the following:

$$\begin{aligned}
\mathrm{sgn}(\sigma') &= \mathrm{sgn}(\sigma'') \\
&= (-1)^{(n-1)-(r-1)} \\
&= (-1)^{n-r} \\
&= \mathrm{sgn}(\sigma).
\end{aligned}$$

■

Proposition 3 (Exercise 2.23)**.** *Show that an r-cycle is an even permutation if and only if r is odd.*

Proof. Let $\sigma \in S_n$ be an r-cycle and let t be the number of disjoint cycles in a complete factorization of σ. Observe that $t \in \{1, ..., n\}$.

First we show that $r = (n-t) + 1$. Since σ is an r-cycle, σ has a complete factorization comprised of σ and any 1-cycles of elements fixed by σ. Since every element of $\{1, ..., n\}$ must appear in one and only one disjoint cycle in a complete factorization, the sum of the lengths of each disjoint cycle must be n. We know that t is the number of cycles in the complete factorization, so $t-1$ is the number of 1-cycles (after removing r), each of which has length 1. Thus,

$$n = r + (t-1)$$

Then after solving for r,

$$r = n - (t-1) = (n-t) + 1. \tag{1}$$

So, our claim is proven.

We observe that r is odd if and only if $(n-t)$ is even; and we can also see that, $n-t$ is even if and only if $(-1)^{n-t} = 1$. Thus, we conclude that the proposition holds.

■

Proposition 4 (Exercise 2.29)**.** *If $n \geq 2$, prove that the number of even permutations in S_n is $\frac{1}{2}n!$.*

Proof. Using the counting principle, it can be seen that S_n has $n!$ many permutations. This follows from the fact that each permutation is one way of permuting the elements in a set with n elements.

We observe that, by the definition of the sgn function, every permutation in S_n is either even or odd. Thus, if we can demonstrate that there exists a bijection of the even permutations onto the odd permutations then not only do the set of even permutations and the set of odd permutations have the same cardinality, it also shows that each has cardinality $\frac{1}{2}n!$.

We define S_n^e to be the set of even permutations in S_n and S_n^o to be the set of odd permutations in S_n. Take any transposition $\tau \in S_n$. We claim that $F : S_n^e \to S_n^o$ defined by the rule $F(\sigma) = \sigma\tau$ is such a bijection.

First, fix $\sigma \in S_n^e$. By Lemma 2.38, $F(\sigma) \in S_n^o$. This shows that F is well defined.

Second, we show that F is bijective by showing that F has an inverse. Since every permutation $\{1, ..., n\} \to \{1, ..., n\}$ is naturally a bijection, then every permutation has an inverse. In particular, τ has an inverse, say τ^{-1}. Define $G : S_n^o \to S_n^e$ by the rule $G(\sigma) = \sigma\tau^{-1}$. We see that G is also well defined by Lemma 2.38. Thus, we must only show that G is a left and a right inverse of F.

Observe that

$$(F \circ G)(\sigma) = F(G(\sigma)) = F(\sigma\tau^{-1}) = (\sigma\tau^{-1})\tau = \sigma.$$

Similarly,

$$(G \circ F)(\sigma) = G(F(\sigma)) = G(\sigma\tau) = (\sigma\tau)\tau^{-1} = \sigma.$$

Thus, $G = F^{-1}$. We conclude that F is the desired bijection and the proposition holds. ■

Solutions to Non-text Problem

Proposition 5 (Exercise A)**.** *Let $\sigma = \sigma_1 \cdots \sigma_t$ be a product decomposition of σ into disjoint cycles. Suppose σ_i is a k_i cycle, and*

$$k = lcm(k_1, ..., k_t).$$

Then k is the smallest positive integer for which $\sigma_k = (1)$.

Proof. We observe that disjoint cycles commute. Thus,

$$\sigma^k = (\sigma_1 \cdots \sigma_t)^k = \sigma_1^k \cdots \sigma_t^k.$$

Since k is the least common multiple of $k_1, ..., k_t$, it follows that for each k_i there exists some m_i such that $k_i m_i = k$ where $1 \le i \le k$. Then we have the following:

$$\sigma^k = (\sigma_1^{k_1})^{m_1} \cdots (\sigma_t^{k_t})^{m_t}.$$

By exercise 2.22, $\sigma_i^{k_i} = (1)$ for each i. Thus, $\sigma^k = (1)$, from what's above. Note that $k \ne 0$, because cycles have at least length 1. Moreover, k is, by definition, the smallest positive common multiple of $k_1, ..., k_t$.

So, there is no smaller multiple m satisfying $\sigma^m = (1)$. For any integer z between

$$1 \leq z \leq k$$

, we see that there exists some k_i for which z is not a multiple of k_i. Thus, $\sigma_i^z \neq (1)$. Thus, there is no such integer z. So, the result holds. ■