HW 2 Selected Solutions

Prof. Shahed Sharif

1.73 We use the fact that $10 \equiv -1 \pmod{11}$, so $10^n \equiv (-1)^n \pmod{11}$. Let n have base 10 expansion $d_r d_{r-1} \ldots d_0$. This means that

$$n = d_0 + 10d_1 + 10^2 d_2 + \cdots + 10^r d_r = \sum_{i=0}^{r} 10^i d_i.$$

Taking the whole expression mod 11, and using $10^n \equiv (-1)^n \pmod{11}$, we obtain

$$n \equiv d_0 - d_1 + d_2 - \cdots + (-1)^r d_r = \sum_{i=0}^{r} (-1)^i d_i,$$

as required.

2.13 For the first, suppose $(g \circ f)(x) = (g \circ f)(y)$. Then $g(f(x)) = g(f(y))$, and since g is injective, $f(x) = f(y)$. But f is injective, so $x = y$. Therefore $g \circ f$ is injective.

For the second, let $z \in Z$. Since g is surjective, $\exists y \in Y$ such that $g(y) = z$. Since f is surjective, $\exists x \in X$ such that $f(x) = y$. In particular, $(g \circ f)(x) = z$. Therefore $g \circ f$ is surjective.

The third is obtained just by combining the first two parts.

For the fourth part, let $z \in Z$. Since $g \circ f$ is surjective, $\exists x \in X$ such that $(g \circ f)(x) = z$. In particular, $g(f(x)) = z$, showing g is surjective. Now suppose $f(x) = f(y)$. Then $(g(f(x))) = g(f(y))$, or $(g \circ f)(x) = (g \circ f)(y)$. Since $g \circ f$ is injective, $x = y$. It follows that f is injective.

A. Let f be such a function. For $f(1)$, there are 4 possibilities; for $f(2)$, there are 4; etc. By the multiplication principle, there are 4^4 functions.