HW 10 Selected solutions

Prof. Shahed Sharif

3.3 For (i), $(3-2)-1 = 1-1 = 0$, while $3-(2-1) = 3-1 = 2$. For (ii), $\mathbb{Z}/2\mathbb{Z}$ works, since in that ring, $+$ and $-$ mean the same thing!

3.4 Use $R = \mathbb{Z}$.

3.5 Straightforward.

3.7 Part (i) is straightforward. For part (ii), the function does not have to be that nice.

A. Define $\varphi : G \to H$ be given by $\varphi(x) = x^2$. We have

$$\begin{aligned}\varphi(xy) &= (xy)^2 \\ &= x^2y^2 \\ &= \varphi(x)\varphi(y),\end{aligned}$$

so φ is a homomorphism. Every positive real number has a real square root, so φ is surjective. As $\varphi(1) = 1^2 = 1$ and $\varphi(-1) = (-1)^2 = 1$, we have $\{\pm 1\} \subset \ker(\varphi)$. On the other hand, if $\varphi(x) = 1$, then $x^2 = 1$, so $x = \pm 1$. Therefore the kernel is exactly $\{1, -1\}$. The claim now follows from the 1st Isomorphism Theorem.

B. Let $\varphi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be given by $\varphi(x, y) = 8x - 5y$. We first show that φ is a homomorphisms. Let $(x, y), (w, z) \in \mathbb{Z} \times \mathbb{Z}$. Then

$$\begin{aligned}\varphi(x+w, y+z) &= 8(x+w) - 5(y+z) \\ &= 8x - 5y + 8w - 5z \\ &= \varphi(x, y) + \varphi(w, z).\end{aligned}$$

Next, the image is the set of linear combinations of 8 and 5, which is the set of multiples of $\gcd(8, 5)$. But $\gcd(8, 5) = 1$, and therefore φ is surjective. More concretely, for $n \in \mathbb{Z}$, $\varphi(2n, 3n) = 16n - 15n = n$.

Next we compute $\ker\varphi$. We have $\varphi(5, 8) = 8 \cdot 5 - 5 \cdot 8 = 0$, so $(5, 8) \in \ker\varphi$, and hence $\langle(5, 8)\rangle \subset \ker\varphi$. Now suppose $(x, y) \in \ker(\varphi)$. This means that $8x - 5y = 0$, or $8x = 5y$. We have $5 \mid 5y$, hence $5 \mid 8x$. But $5 \nmid 8$, so by Euclid's Lemma, $5 \mid x$. Thus we may write $x = 5n$. Substituting, we get $40n = 5y$, and so $y = 8n$. This shows that $(x, y) = (5n, 8n) \in \langle(5, 8)\rangle$. Therefore $\ker(\varphi) = \langle(5, 8)\rangle$.

Finally, we invoke the 1st Isomorphism Theorem, and the claim follows.

C. Define $\varphi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}_2$ by $\varphi(x, y) = (3x - 8y, [y])$, where $[y]$ means the class of y mod 2. I omit the proof that this is a homomorphism; it is similar to B. Observe that $\varphi(-5, -2) = (1, 0)$ and $\varphi(8, 3) = (0, 1)$. It follows that

$$\varphi(a(-5, -2) + b(8, 3)) = (a, b),$$

and so φ is surjective.

Now we compute $\ker(\varphi)$. We have that $\varphi(16, 6) = (3 \cdot 16 - 8 \cdot 6, [6]) = (0, 0)$, so $(16, 6) \in \ker(\varphi)$. Since the kernel is a subgroup, this implies that $\langle (16, 6) \rangle \subset \ker(\varphi)$. Now suppose $(x, y) \in \ker(\varphi)$. Then $3x - 8y = 0$ and $2 \mid y$. We have $3x = 8y$, so $3 \mid 8y$, and therefore by Euclid's Lemma, $3 \mid y$. But also $2 \mid y$, and hence $6 \mid y$. Thus we may write $y = 6n$. Therefore $3x = 48n$, and so $x = 16n$. Thus $(x, y) = (16n, 6n) \in \langle (16, 6) \rangle$. We conclude that $\ker(\varphi) = \langle (16, 6) \rangle$.

The result now follows from the 1st Isomorphism Theorem.

D. Use the definitions.

E. Use the definitions, and that $\exists g$ with $gx = y$.

F. Do out a couple examples. For instance, take the sequence given by $a_n = n$ ($n \geq 1$), and pick various g.