HW 1 Selected Solutions

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1.17 We do this by strong induction. The cases of F_0 and F_1 are easy. (The inductive step required knowing the previous 2 cases, so the base case needs to do the first 2 cases.) For the inductive step, assume that for $k < n$, $F_k < 2^k$. Then

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} \\ &< 2^{n-1} + 2^{n-2} \\ &< 2^{n-1} + 2^{n-1} \\ &= 2^n. \end{aligned}$$

By induction, the claim holds.

1.47 I will show existence by strong induction. The base case is $m = 1$, in which case we have $1 = 2^0$. For the inductive step, suppose every $m \geq 2$ and for ever $1 \leq k < m$, k can be written as a sum of distinct powers of 2. Since $2^0 < m$, $\lim_{n\to\infty} 2^n = \infty$, and 2^n is strictly increasing, there is some n such that $2^n \leq m < 2^{n+1}$. If $m = 2^n$, then we are done. Otherwise, let $k = m - 2^n$, so that $1 \leq k < m$. By our inductive hypothesis, we can write k as a distinct sum of powers of 2; say,

$$k = 2^{e_1} + 2^{e_2} + \cdots + 2^{e_r}.$$

Without loss of generality, $e_1 < e_2 < \cdots < e_r$. Thus

$$m = 2^{e_1} + 2^{e_2} + \cdots + 2^{e_r} + 2^n.$$

It remains to show that $n \neq e_i$ for all i. But if we had $e_i = e_n$ for some i, then we would get

$$\begin{aligned} m &= 2^{e_1} + 2^{e_2} + \cdots + 2^{e_r} + 2^n \\ &\geq 2^{e_i} + 2^n \\ &= 2^n + 2^n \\ &= 2^{n+1}. \end{aligned}$$

In particular, $m \geq 2^{n+1}$, which contradicts our choice of n. Therefore m can be written as a sum of distinct powers of 2.

There is a cute alternative method of proving the statement *without* induction, which I skethch out. One first shows uniqueness. Then choose m, find n so that $2^{n+1} > m$, and consider the set B_n of all numbers of the form

$$a_0 + a_1 2^1 + a_2 2^2 + \cdots + a_n 2^n$$

with $a_i = 0$ or 1 for all i. By the uniqueness, these numbers are all distinct. The smallest number of this form is 0, and the largest, when all of the a_i are 1, is $2^{n+1} - 1$. There are exactly 2^{n+1} numbers in this range. But by the Multiplication Principle, $\#B_n = 2^{n+1}$, and so every number in the range must occur. In particular, m must be in B_n, and so can be written as a sum of distinct powers of 2.

Finally, there is a quick, somewhat cheesy (but correct) way of doing this problem: just reference Prop. 1.44 and observe that the d_i must be 0 or 1.

1.54 Let $s_k = s + kb$ and $t_k = t - ka$. Since $(a, b) \neq (0, 0)$, the pairs (s_k, t_k) are all distinct. Finally, we have

$$\begin{aligned} s_k a + t_k b &= (s + kb)a + (t - ka)b \\ &= sa + tb + kba - kab \\ &= sa + tb \\ &= d. \end{aligned}$$