Name:

Math 470: Final Exam

December 11, 2023

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. You may use any result in the relevant sections of the textbook, or from lecture, but may not use homework problems. Avoid using the back of the page; instead, there is an extra sheet at the end that you can use. Calculators are not allowed.

1. (a) (10 points) In S_7, compute the order of $(1\,2\,6\,4\,5)(2\,4)$.

(a) ______________

Solution: The product is $(1\,2\,6)(4\,5)$ which has order 6.

(b) (5 points) Let $G = S_4$ and $H = \langle(1\,3)(2\,4)\rangle$. Compute the index $[G : H]$.

(b) ______________

Solution: By Lagrange's Theorem, $[G : H] = \#G/\#H = 4!/2 = 12$.

(c) (10 points) Give an example of a group homomorphism which is not a ring homomorphism. Justify your answer.

Solution: Let $\varphi : \mathbb{Z}_2 \to \mathbb{Z}_4$ be given by $\varphi(0) = 0$ and $\varphi(1) = 2$. Since $\langle 2\rangle \subset \mathbb{Z}_4$ is a subgroup of order 2, it must by cyclic, and hence is isomorphic to $\mathbb{Z}_2$; φ is that isomorphism. But this is not a ring homomorphism, since $\varphi(1^2) = \varphi(1) = 2$, while $\varphi(1)^2 = 2^2 = 0$, so $\varphi(1^2) \neq \varphi(1)^2$.

(d) (5 points) Show that there is a commutative ring R and a nonzero polynomial $f(x) \in R[x]$ of degree n with more than n roots in R.

Solution: Again, there are many solutions. The one I gave in class is $x^2 - 1 \in \mathbb{Z}_8[x]$. For this polynomial, $1, 3, 5, 7$ are all roots.

(e) (5 points) Define PID.

Solution: A domain in which every ideal is principal.

2. (15 points) Consider $\mathbb{R}\backslash\{-1\}$ equipped with the binary operation $*$ defined by

$$a * b = ab + a + b.$$

Prove that $(\mathbb{R}\backslash\{-1\}, *)$ is a group.

Solution: This is well-defined; that is, we cannot have $a * b = -1$, for then we'd have $a, b \neq 1$ satisfying $ab + a + b = -1$. This implies that $a(b + 1) = -(b + 1)$. This can only occur if either $a = -1$ or $b = -1$, and neither can hold.

Next, we show associativity. Given $a, b, c \neq 1$, we have

$$\begin{aligned} a * (b * c) &= a * (bc + b + c) \\ &= a(bc + b + c) + a + (bc + b + c) \\ &= abc + ab + ac + a + bc + b + c \\ &= abc + ac + bc + ab + a + b + c \\ &= (ab + a + b)c + (ab + a + b) + c \\ &= (a * b) * c. \end{aligned}$$

Next, I claim that 0 is the identity. We have for $a \neq -1$,

$$\begin{aligned} a * 0 &= a \cdot 0 + a + 0 \\ &= a. \end{aligned}$$

This proves the claim.

Finally, I have to show existence of inverses. Given $a \neq -1$, I claim that $b = -\frac{a}{a+1}$ is the inverse of a. For

$$\begin{aligned} a * b &= a * \frac{-a}{a+1} \\ &= \frac{-a^2}{a+1} + a - \frac{a}{a+1} \\ &= \frac{-a^2 + a^2 + a - a}{a+1} \\ &= 0. \end{aligned}$$

This shows that $b = a^{-1}$.

3. (10 points) Suppose G is a group for which $x^2 = e$ for all $x \in G$. Prove that G is abelian.

Solution: Let $a, b \in G$. We have $(ab)^2 = e$, so $abab = e$. Multiplying both sides on the left by a and on the right by b, we obtain $a^2bab^2 = ab$. But $a^2 = b^2 = e$, so $ba = ab$. This holds for all $a, b \in G$, and so G is abelian.

4. Let $G = \mathbb{Z}_4 \times \mathbb{Z}_4$ and $H = \langle(1, 2)\rangle$.
   (a) (5 points) Compute $\#H$.

(a) ______________

Solution: The order is 4; the elements are

$$(0, 0), (1, 2), (2, 0), (3, 2).$$

   (b) (5 points) Compute $[G : H]$.

(b) ______________

Solution: It is $\#G/\#H = 16/4 = 4$.

(c) (10 points) Find a set of coset representatives for G/H.

Solution: These are (for instance) $(0,0),(0,1),(0,2),(0,3)$. Since there are 4 cosets, it suffices to show that no pair of these is in the same coset. But if $(0,i)$ and $(0,j)$ are in the same coset, then $(0,i-j) \in H$. But looking at our list of elements above, this can only hold if $i-j=0$; in other words, if $i=j$. Thus no two different elements are in the same coset, as required.

5. (15 points) Show that $\mathbb{Z}_{11}^\times \cong \mathbb{Z}_{10}$.

Solution: Define $\varphi : \mathbb{Z} \to \mathbb{Z}_{11}^\times$ by $\varphi(n) = 2^n$. Clearly, φ is a homomorphism. The key fact here is that 2 has order 10 in $\mathbb{Z}_{11}^\times$; this can be verified by listing the powers of 2 mod 11 (omitted). In other words, $\mathbb{Z}_{11}^\times = \langle 2 \rangle$. It follows that φ is surjective. The kernel is the set of n for which $2^n = 1$; in other words, the set of periods of 2. But since the order of 2 is 10, the set of periods is $10\mathbb{Z}$. By the First Isomorphism Theorem, we get an isomorphism $\mathbb{Z}_{10} \to \mathbb{Z}_{11}^\times$ as required.

6. (15 points) Show that $\mathbb{Z}^2/\langle(3,5)\rangle \cong \mathbb{Z}$.

Solution: Define $\varphi : \mathbb{Z}^2 \to \mathbb{Z}$ by $\varphi(x,y) = 5x - 3y$. This is a homomorphism: for $x,y,u,v \in \mathbb{Z}$,

$$\begin{aligned}
\varphi(x,y) + \varphi(u,v) &= 5x - 3y + 5u - 3v \\
&= 5(x+u) - 3(y+v) \\
&= \varphi(x+u, y+v) \\
&= \varphi((x,y) + (u,v)).
\end{aligned}$$

The image is the set of linear combinations of 5 and 3, which is the set of multiples of $\gcd(5,3)$. But the gcd is 1, so φ is surjective.

Next, we compute $\ker(\varphi)$. As $\varphi(3,5) = 15 - 15 = 0$, $(3,5) \in \ker(\varphi)$. Since $\ker(\varphi)$ is a subgroup, this means that $\langle(3,5)\rangle \subset \ker(\varphi)$. Conversely, suppose $(x,y) \in \ker(\varphi)$. Then $\varphi(x,y) = 0$, so $5x - 3y = 0$, or $5x = 3y$. By Euclid's Lemma, $3 \mid 5x$ implies $3 \mid x$. Thus $x = 3k$ for some k. But then $15k = 3y$, so $y = 5k$. Thus $(x,y) = (3k,5k) = (3,5)k \in \langle(3,5)\rangle$. Therefore $\ker(\varphi) = \langle(3,5)\rangle$.

The result now follows from the 1st Isomorphism Theorem.

7. (15 points) Let T be the group of symmetries (rotations *and* reflections) of a triangular prism. The base of the prism is an equilateral triangle, and the sides are rectangles. What is $\#T$?

Solution: We use the Orbit-Stabilizer Theorem. As usual, there are many ways to proceed. Let's pick a vertex. By rotating along the central axis (horizontal in the picture), we can send that vertex to 3 vertices. By flipping the prism around and rotating about the central axis, we can send that vertex to the other 3 vertices. Therefore the orbit size is 6.

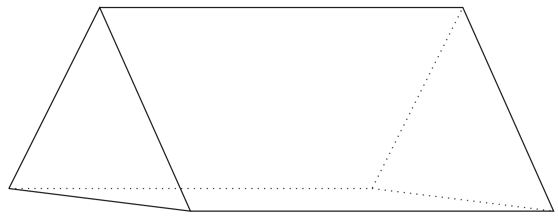

Figure 1: A triangular prism

Next we compute the stabilizer. For a given vertex, there are only 2 symmetries that fix it: the identity, and a reflection. The plane of the reflection passes through the vertex, is perpendicular to the triangular faces, and divides each triangle into two equal pieces.

By the Orbit-Stabilizer Theorem, there are $6 \cdot 2 = 12$ symmetries total.

Another natural choice is to pick one of the triangular faces. If you do that, the orbit size is 2 (for the two triangles) and the stabilizer size is 6 (for the elements of D_6, each of which extends to a symmetry of the prism).

8. (15 points) Show that if R is a unital commutative ring, then the set of units in R forms a group under multiplication.

Solution: Let $R^\times$ be the set of units. As R is a ring, multiplication in R is associative, and hence the same is true in $R^\times$. Since R is unital, $1 \in R$, and certainly 1 is a unit ($1 \cdot 1 = 1$). Hence $R^\times$ has an identity. If $u \in R^\times$, then by definition of $R^\times$, $\exists v \in R$ such that $uv = vu = 1$. By symmetry, the inverse of v is u, so we also have $v \in R^\times$. Thus $R^\times$ is closed under inversion. Lastly, if $u, w \in R^\times$, then $(uw)(w^{-1}u^{-1}) = 1$, and hence $uw \in R^\times$. Thus $R^\times$ is closed under multiplication.

Note that commutativity is unnecessary. I only included it to make closure under inversion a little easier to prove.