Name:

Math 470: Exam 6

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. Calculators are not allowed.

You may use any result from the chapters covered in the text or from lecture. You may not use the results of homework or worksheet problems.

1. (10 pts) Show that $\langle (1\,2\,3\,4) \rangle$ is not a normal subgroup of S_4.

Solution: Taking powers of the generator, we have

$$\langle (1\,2\,3\,4) \rangle = \{(1), (1\,2\,3\,4), (1\,3)(2\,4), (4\,3\,2\,1)\}.$$

Next,

$$(1\,2)(1\,2\,3\,4)(1\,2)^{-1} = (2\,1\,3\,4)$$

using our characterization of conjugation in S_n. But this latter is not in the subgroup. Therefore the subgroup is not normal.

2. (20 pts) Let σ be the usual element of D_6. Prove that the map

$$\begin{aligned} \varphi : \mathbb{Z}/3\mathbb{Z} &\to \langle \sigma \rangle \\ [i] &\mapsto \sigma^i \end{aligned}$$

is a well-defined isomorphism.

Solution: Recall that $\sigma^3 = e$.

Suppose $[i] = [j]$. That means $i \equiv j \pmod 3$, so $\exists k \in \mathbb{Z}$ such that $i = j + 3k$. Then

$$\begin{aligned} \varphi([i]) &= \sigma^i \\ &= \sigma^{j+3k} \\ &= \sigma^j (\sigma^3)^k \\ &= \sigma^j \\ &= \varphi([j]). \end{aligned}$$

Thus φ is well-defined.

We have

$$\begin{aligned} \varphi([a] + [b]) &= \varphi([a + b]) \\ &= \sigma^{a+b} \\ &= \sigma^a \sigma^b \\ &= \varphi([a]) + \varphi([b]). \end{aligned}$$

Thus φ is a homomorphism.

Finally, $\varphi([0]) = e$, $\varphi([1]) = \sigma$, $\varphi([2]) = \sigma^2$, so this map is visibly bijective.

3. (20 pts) Show that $\mathbb{Z}/12\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ are not isomorphic.

Solution: Let $\varphi : \mathbb{Z}/12\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ be a function. Suppose for the sake of contradiction that φ is an isomorphism. Then the order of $\varphi(1)$ should be the order of 1, which is 12. Say $\varphi(1) = (a, b)$. Then

$$\begin{aligned} 6(a, b) &= (6a, 6b) \\ &= (0, 0) \end{aligned}$$

since $6 \mid 6a$ and $2 \mid 6b$. Thus (a, b) has period 6. As $6 < 12$, the order cannot be 12. This is the desired contradiction. Therefore there is no isomorphism between the two groups.