Name:

Math 470: Exam 3

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. Calculators are not allowed.

You may use any result from the chapters covered in the text or from lecture. You may not use the results of homework or worksheet problems.

1. (5 pts) Compute $\operatorname{sgn}(\sigma)$ where $\sigma = (1\,2\,3\,4)(4\,6)(5\,8\,6)$.

Solution: We have $\operatorname{sgn}(1\,2\,3\,4) = -1$, $\operatorname{sgn}(4\,6) = -1$, and $\operatorname{sgn}(5\,8\,6) = +1$. Therefore $\operatorname{sgn}(\sigma) = (-1)(-1)(+1) = +1$.

2. (5 pts) Give an example of an abelian group that has 6 elements.

Solution: $\mathbb{Z}/6\mathbb{Z}$ under addition.

3. (5 pts) Give an example of a nonabelian group with infinitely many elements.

Solution: $\mathrm{GL}_2(\mathbb{R})$. Alternatively S_X for any infinite set X of your choice.

4. (15 pts) Suppose G is a group and $g \in G$ has order 11. Prove that g^2 also has order 11.

Solution: We have

$$\begin{aligned}(g^2)^{11} &= (g^{11})^2 \\ &= e^2 = e,\end{aligned}$$

so 11 is a period of g. The order divides the period, so the order is either 11 as required, or 1. If the order were 1, then we'd have $g^2 = e$, which contradicts that the order of g is 11. The claim follows.

5. (20 pts) Consider the set $\mathbb{R}\backslash\{0\}$ with the binary operation $a * b = \frac{ab}{2}$. Prove that $(\mathbb{R}\backslash\{0\}, *)$ forms a group.

Solution: The product of nonzero reals is a nonzero real, so the set is closed under $*$.

Let $e = 2$. Then $a * 2 = \frac{2a}{2} = a$. Thus there is an identity.

Given a, let $a^{-1} = 4/a$, which is certainly in $\mathbb{R}\backslash\{0\}$. Observe that

$$\begin{aligned}a * \frac{4}{a} &= \frac{a \cdot \frac{4}{a}}{2} \\ &= \frac{4}{2} \\ &= 2 = e,\end{aligned}$$

so inverses exist.

Lastly, given $a, b, c \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned}
(a * b) * c &= \frac{ab}{2} * c \\
&= \frac{\frac{ab}{2}c}{2} \\
&= \frac{abc}{4} \\
&= \frac{a\frac{bc}{2}}{2} \\
&= a * \frac{bc}{2} \\
&= a * (b * c).
\end{aligned}$$

Associativity follows.