Name:

Math 470: Exam 2

November 13, 2023

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. You may use any result in the relevant sections of the textbook, or from lecture, but may not use homework problems. Avoid using the back of the page; instead, there is an extra sheet at the end that you can use. Calculators are not allowed.

1. (a) (5 points) Let $G = S_5$ and $H = \langle (1\,2\,3) \rangle$. Compute the index $[G : H]$.

(a) ________________

Solution: By the proof of Lagrange's Theorem, $[G : H] = \#G/\#H = 5!/3 = 5{\cdot}4{\cdot}2{\cdot}1 = 40$.

(b) (5 points) Give an example of two nonisomorphic groups of order 27. You do not have to prove your answer.

(b) ________________

Solution: $\mathbb{Z}_{27}$ and $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$

(c) (5 points) Give an example of a subgroup of D_6 which is not normal. You do not have to prove your answer.

(c) ________________

Solution: $H = \langle \tau \rangle$ is not normal.

(d) (5 points) Give an example of a homomorphism $\varphi : \mathbb{R}^\times \to \mathbb{R}^\times$ with kernel of size 2.

(d) ________________

Solution: $\varphi(x) = x^2$ or $\varphi(x) = |x|$ both work. The kernel in both cases is $\{1, -1\}$.

2. (10 points) Suppose $\varphi : D_8 \to S_4$ is a homomorphism satisfying $\varphi(\sigma) = (1\,2\,3\,4)$ and $\varphi(\tau) = (1\,2)(3\,4)$. Compute $\varphi(\tau\sigma)$ and $\varphi(\sigma^3\tau)$.

Solution: We have

$$\begin{aligned}\varphi(\tau\sigma) &= \varphi(\tau)\varphi(\sigma)\\ &= (1\,2)(3\,4)(1\,2\,3\,4)\\ &= (2\,4).\end{aligned}$$

Since $\sigma^3\tau = \tau\sigma$, $\varphi(\sigma^3\tau) = (2\,4)$ also.

3. (15 points) Show that $\mathbb{Z}_{10}^\times \cong \mathbb{Z}_4$.

Solution: Define $\varphi : \mathbb{Z}_4 \to \mathbb{Z}_{10}^\times$ by $\varphi(x) = 3^x$. Observe that $3^4 = 81 \equiv 1 \pmod{10}$, while $3^1 \not\equiv 1 \pmod{10}$ and $3^2 \equiv 9 \not\equiv 1 \pmod{10}$. Thus 3 has order 4. This means that φ is well-defined: if $x \equiv y \pmod 4$, then $y = x + 4k$ for some $k \in \mathbb{Z}$, and

$$\varphi(y) = 3^y = 3^{x+4k} = 3^x(3^4)^k = 3^x = \varphi(x).$$

Since the order of 3 is 4, φ is injective. Finally, $\mathbb{Z}_{10}^\times = \{1, 3, 7, 9\}$ has 4 elements, so by the Pigeonhole Principle, φ is surjective.

4. Let S_n be the group of permutations of the set $\{1, 2, \ldots, n\}$ with $n \geq 3$. Let

$$H = \{\sigma \in S_n : \sigma(1) = 1\}.$$

(a) (15 points) Prove that H is a subgroup.

Solution: To show closure under multiplication, let $\sigma, \tau \in H$. Then

$$\begin{aligned}(\sigma\tau)(1) &= \sigma(\tau(1)) \\ &= \sigma(1) \\ &= 1.\end{aligned}$$

Each equality depends only on the definition of H. Again by the definition of H, $\sigma\tau \in H$.

Now suppose $\sigma \in H$. Then

$$\begin{aligned}(\sigma^{-1}\sigma)(1) &= \sigma^{-1}(\sigma(1)) \\ &= \sigma^{-1}(1).\end{aligned}$$

On the other hand, $(\sigma^{-1}\sigma)(1) = id(1) = 1$, where id denotes the identity permutation. Therefore $\sigma^{-1}(1) = 1$ and $\sigma^{-1} \in H$.

(b) (5 points) Prove that H is not a normal subgroup.

Solution: Observe that $(2\,3) \in H$. Now

$$(1\,3)(2\,3)(1\,3)^{-1} = (2\,1) \notin H$$

which proves the claim.

5. Let $H = \langle \sigma^2 \rangle \subset D_8$.

(a) (10 points) Show that $H \lhd D_8$. Remember to show that it is a subgroup!

Solution: Cyclic groups are always subgroups! For normality, observe that $H = \{e, \sigma^2\}$ since $(\sigma^2)^2 = \sigma^4 = e$, so it suffices to show that $\alpha\sigma^2\alpha^{-1} \in H$ for all $\alpha \in D_8$. There are 2 cases: $\alpha = \sigma^i$ for some i, or $\alpha = \sigma^i\tau$ for some i.

In the first case,

$$\begin{aligned}\alpha\sigma^2\alpha^{-1} &= \sigma^i\sigma^2\sigma^{-i} \\ &= \sigma^{i+2-i} \\ &= \sigma^2 \in H.\end{aligned}$$

For the second case, observing that $\tau^{-1} = \tau$ and $\tau\sigma^{-i} = \sigma^i\tau$,

$$\begin{aligned}\alpha\sigma^2\alpha^{-1} &= \sigma^i\tau\sigma^2(\sigma^i\tau)^{-1} \\ &= \sigma^i\tau\sigma^2\tau\sigma^{-i} \\ &= \sigma^i\tau\sigma^2\sigma^i\tau \\ &= \sigma^i\sigma^{-i-2}\tau\tau \\ &= \sigma^{i-i-2} \\ &= \sigma^{-2} \\ &= \sigma^2 \in H\end{aligned}$$

where $\sigma^{-2} = \sigma^2$ since $\sigma^4 = e$. The claim follows.

(b) (15 points) Find a complete set of coset representatives, and compute the products of all pairs of coset representatives. Every answer should be one of your representatives! You may use a multiplication table to summarize your answer.

Solution: As $\#D_8/\#H = 8/2 = 4$, there are 4 cosets. One checks that $e, \sigma, \tau, \sigma\tau$ are representatives.
We have the following multiplication table:

	e	σ	τ	$\sigma\tau$
e	e	σ	τ	$\sigma\tau$
σ	σ	e	$\sigma\tau$	τ
τ	τ	$\sigma\tau$	e	σ
$\sigma\tau$	$\sigma\tau$	τ	σ	e

To see this, note that $\sigma \cdot \sigma\tau = \sigma^2\tau \in H\tau = \tau H$ since H is normal, and hence we get the representative τ. Or take $\tau \cdot \sigma = \sigma^3\tau = \sigma^2\sigma\tau \in H\sigma\tau = \sigma\tau H$. Thus we get $\sigma\tau$ in the corresponding entry of the table. The rest are obtained similarly.

6. (15 points) Let $\varphi : G \to K$ be a surjective homomorphism of finite groups, and let $H = \ker(\varphi)$. Let $k \in K$. Show that the cardinalities $\#\varphi^{-1}(k)$ and $\#H$ are equal. (You may not use the 1st Isomorphism Theorem.)

Solution: I will construct a bijection $H \to \varphi^{-1}(k)$. Let $a \in \varphi^{-1}(k)$. Then define

$$f : H \to \varphi^{-1}(k)$$

by $f(h) = ah$. This is well-defined, in the sense that the image lies in $\varphi^{-1}(k)$:

$$\begin{aligned}\varphi(ah) &= \varphi(a)\varphi(h) \\ &= k \cdot e \\ &= k.\end{aligned}$$

Thus $ah \in \varphi^{-1}(k)$. It is injective: suppose $f(h_1) = f(h_2)$. Then $ah_1 = ah_2$, and multiplying by a^{-1} on the left yields $h_1 = h_2$. Finally for surjectivity, let $b \in \varphi^{-1}(k)$. Set $h = a^{-1}b$.

Observe that $\varphi(a) = \varphi(b) = k$, so

$$\begin{aligned} \varphi(h) &= \varphi(a^{-1}b) \\ &= \varphi(a)^{-1}\varphi(b) \\ &= k^{-1}k \\ &= e, \end{aligned}$$

and therefore $h \in \ker(\varphi) = H$. I claim that $f(h) = b$. For

$$\begin{aligned} f(h) &= ah \\ &= aa^{-1}b \\ &= b. \end{aligned}$$

Thus f is a bijection.

The claim follows.