Name:

Math 470: Exam 1

October 4, 2023

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. Avoid using the back of the page; instead, there is an extra sheet at the end that you can use. Calculators are not allowed.

1. (10 points) Find $\gcd(775, 387)$.

1. ______________

> **Solution:** We have $775 = 2 \cdot 387 + 1$, so $\gcd(775, 387) = \gcd(387, 1) = 1$.

2. (10 points) If n is odd, prove that $n^2 \equiv 1 \pmod 4$.

> **Solution:** If n is odd, then $n \equiv 1$ or $3 \pmod 4$. In either case, squaring both sides yields $n^2 \equiv 1 \pmod 4$. Alternatively, write $n = 2k + 1$ and square.

3. Let X, Y, Z be sets and $f : X \to Y$, $g : Y \to Z$ be functions. Suppose that $g \circ f$ is a bijection.
   (a) (10 points) Prove that f is an injection.
   (b) (10 points) Prove that g is a surjection.
   (c) (10 points) True or false: both f and g must in fact be bijections. Prove your answer if true, and give a counterexample if false.

> **Solution:** All three were homework problems; see those solutions.

4. Let $\alpha = (1\,2\,3\,5)(2\,4\,5\,6\,7)(1\,8\,7\,2)(2\,9\,4\,6)$ be an element of S_{11}.
   (a) (10 points) Write α as a product of disjoint cycles.

> **Solution:** $(1\,8\,3\,5\,6\,2\,9)(4\,7)$

   (b) (5 points) Compute the inverse of α, as a product of disjoint cycles.

> **Solution:** $(9\,2\,6\,5\,3\,8\,1)(7\,4)$ (they are disjoint, hence commute, hence the order of the two cycles doesn't matter)

   (c) (5 points) Compute $\operatorname{sgn}(\alpha)$.

> **Solution:** The first cycle has length 7, so is even. The second is odd. Therefore the product is odd, so $\operatorname{sgn}(\alpha) = -1$.

5. (10 points) Let $\sigma, \tau \in S_n$ be

$$\sigma = (1 \quad 2 \quad \cdots \quad n)$$
$$\tau = (1 \quad 2)$$

Prove that $(i \; i+1)$ can be written as a product of powers of σ and τ for any $1 \leq i \leq n-1$.

Solution: We do this by induction. The base case, $i = 1$ is just τ itself. Now suppose we can get $(i-1 \; i)$ as a product of powers of σ and τ for $2 \leq i \leq n-1$. Observe that $\sigma(i-1) = i$ and $\sigma(i) = i+1$. By a result in the text,

$$\sigma(i-1 \; i)\sigma^{-1} = (i \; i+1).$$

Applying our inductive hypothesis on the transposition on the left, we obtain the conclusion for $(i \; i+1)$. By induction, the claim follows.

6. No proofs necessary for this problem.

   (a) (5 points) Give an example of an abelian group with 6 elements.

   (a) ______________

   (b) (5 points) Give an example of a nonabelian group with 6 elements.

   (b) ______________

   (c) (5 points) Give an example of an infinite nonabelian group.

   (c) ______________

Solution: For the first, $\mathbb{Z}_6$. For the second, D_6 or S_3. For the third, there are many answers; the simplest is probably the set of 2×2 invertible real matrices, also written $\mathrm{GL}_2(\mathbb{R})$.

7. (15 points) Consider $\mathbb{Z}$ under the operation $x * y = x + y + 1$. Prove that $(\mathbb{Z}, *)$ is a group.

Solution: Certainly $\mathbb{Z}$ is closed under $*$. Let $x, y, z \in \mathbb{Z}$. Then

$$\begin{aligned}(x * y) * z &= (x + y + 1) * z \\ &= x + y + 1 + z + 1 \\ &= x + (y + z + 1) + 1 \\ &= x * (y * z).\end{aligned}$$

Thus $*$ is associative.

We have

$$\begin{aligned}x * (-1) &= x - 1 + 1 \\ &= x\end{aligned}$$

and similarly $(-1) * x = x$. Therefore -1 is the identity.

Lastly,

$$\begin{aligned} x * (-2 - x) &= x - 2 - x + 1 \\ &= -1 \end{aligned}$$

so the inverse of x is $-2 - x$.