# HW 1
## Due: Tuesday, February 4

1. Consider the following axiomatic system:

   (a) The population of Smalltown consists exclusively of young married couples and their children.

   (b) There are more adults than children.

   (c) Every boy has a sister.

   (d) There are more boys than girls.

   (e) There are no childless couples.

   Show that the system is inconsistent. Is it meaningful to ask which of these statements is inconsistent?

   **Solution:** Let $c$ be the number of couples, so that there are $2c$ adults by Axiom a. By Axiom e, every couple has a child. I claim that every couple has a girl. For if a couple has a girl, we are done. If they have a boy, then by Axiom c, the boy has a sister, and hence the couple has a girl. Letting $g$ be the number of girls, we have $g \geq c$. By Axiom d, $b > g$. The total number of children is thus

   $$b + g > g + g \geq 2c.$$

   Thus the total number of children exceeds the number of adults, contradicting Axiom b. The system is thus inconsistent.

   It is not meaningful to identify any one of the statements as inconsistent, only the collection.

2. Show that if any one of the five statements in the previous exercise is omitted, the resulting system is consistent. (Note that this is a five part problem!)

   **Solution:** We give a model for each system obtained by omitting one of the axioms. We label the models below by the omitted axiom.

   (a) Take 4 single adults, one of whom has 2 boys and a girl, the rest childless.

   (b) Take one couple with 2 boys and a girl.

   (c) Take one couple with 1 boy.

(d) Take one couple with 1 girl.

(e) Take 2 couples, one of whom has 2 boys and a girl, the other of which is childless.

3. A system consists of a set of elements $S$ and an operation of combination, denoted by the symbol $*$, such that the result of combining a pair of elements of $S$ is also a member of $S$ (that is, if $a, b \in S$, then $a * b \in S$). The system is described by the following axioms:

   (a) If $a, b \in S$, then $a * b = b * a$.

   (b) If $a, b \in S$, then $a * b = b$.

   (c) $S$ contains at least two elements.

   Prove that the system is inconsistent.

**Solution:** By Axiom c, $S$ has at least two different elements; call these $x, y$. By Axiom b, $x * y = y$ and $y * x = x$. By Axiom a, $x * y = y * x$. It follows that $x = y$, which contradicts our choice that the two elements were different.

4. Recall the 6 projective axioms, with model given by the Fano plane. Prove that in the projective axiomatic system, every point lies on at least 3 lines.

**Solution:** Let $P$ be a point. From class, we know that $\exists$ a line $\ell$ not containing $P$. By Axiom 4, $\ell$ contains at least 3 points. Call these $A, B, C$. By Axiom 1, there is a line $\ell_A$ through $P$ and $A$, and similarly lines $\ell_B$ through $P$ and $B$ and $\ell_C$ through $P$ and $C$.

It remains to show that these three lines are distinct. Suppose for the sake of contradiction that $\ell_A = \ell_B$. Then both $A$ and $B$ are on $\ell_A$. Also, both $A$ and $B$ are on $\ell$, so by Axiom 2, $\ell_A = \ell$. But $P \in \ell_A$ while $P \notin \ell$, so this is a contradiction. Therefore $\ell_A \neq \ell_B$. The other two cases are similar.