Name:

Math 410: Exam 2

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. No aids are allowed.

Please do not put any work on the back of pages. Use the space on the last page instead.

1. Let r_1 be reflection in the line $x = 0$ and r_2 reflection in the line $x = 1$.
   (a) (5 pts) Give an algebraic formula for r_1.

   Solution: $r_1(x, y) = (-x, y)$

   (b) (5 pts) Same, for r_2.

   Solution: Let t be translation by $(1, 0)$. Then $r_2 = t \circ r_1 \circ t^{-1}$, so

$$\begin{aligned} r_2(x, y) &= t(r_1(t^{-1}(x, y))) \\ &= t(r_1(x - 1, y)) \\ &= t(1 - x, y) \\ &= (2 - x, y). \end{aligned}$$

   (c) (5 pts) Now give an algebraic formula for $r_1 \circ r_2$.

   Solution: We can do this purely algebraically, or we can observe that reflection in two parallel lines is translation by twice the distance between them, in the perpendicular direction from the first line to the second. The "first" line in this situation is $x = 1$, so we get translation by $(-2, 0)$: $(r_1 \circ r_2)(x, y) = (x - 2, y)$.

2. (10 pts) Let $r : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by $\pi/2$ about the origin. Prove directly from the definition of r and isometry that r is an isometry.

   Solution: We have $r(x, y) = (cx - sy, sx + cy)$ for some c, s. Plugging in $r(1, 0) = (0, 1)$ and $r(0, 1) = (-1, 0)$ yields $r(x, y) = (-y, x)$. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$. Then

$$\begin{aligned} d(r(P), r(Q)) &= d((-y_1, x_1), (-y_2, x_2)) \\ &= \sqrt{(-y_1 - (-y_2))^2 + (x_1 - x_2)^2} \\ &= \sqrt{(y_2 - y_1)^2 + (x_1 - x_2)^2} \\ &= \sqrt{(y_1 - y_2)^2 + (x_1 - x_2)^2} \text{ since } (-1)^2 = 1 \\ &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= d(P, Q). \end{aligned}$$

3. Let $A = (0, 0)$, $B = (1, 0)$, $C = (0, 1)$. Let $A' = (-2, 0)$, $B' = (-2, 1)$, and $C' = (-3, 0)$. Suppose T is an isometry that sends A, B, C to A', B', C', respectively. Write T as a product of up to three reflections. (To specify each reflection, just give its axis of symmetry.)

Solution: We take the first reflection r_1 to take A to A'. The perpendicular bisector is given by $x = -1$. Using this axis, we get

$$r_1(A) = A', \quad r_1(B) = (-3, 0), \quad r_1(C) = (-2, 1).$$

Next we take r_2 to be the reflection taking $(-3, 0)$ to B'. This is the line $y = -(x + 2)$. As this line passes through A', we get $r_2(A') = A'$. But notice also that $r_2(-2, 1) = (-3, 0) = C'$. Thus $r_2 \circ r_1$ is the desired product.

4. Suppose a tetrahedron has vertices (in $\mathbb{R}^3$) $0, \overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}$.
   (a) Compute the centroid of the tetrahedron.

   Solution: It is
   $$\frac{1}{4}(\overrightarrow{u} + \overrightarrow{v} + \overrightarrow{w}).$$

   (b) Show that for each vertex, the line from the vertex to the centroid of the opposite side passes through the centroid of the tetrahedron.

   Solution: By symmetry, it suffice to check the vertex 0. The opposite face has vertices $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}$, so its centroid is
   $$\frac{1}{3}(\overrightarrow{u} + \overrightarrow{v} + \overrightarrow{w}).$$
   The line from 0 to the above centroid is given by
   $$t \cdot \frac{1}{3}(\overrightarrow{u} + \overrightarrow{v} + \overrightarrow{w}) + (1 - t) \cdot 0 = \frac{t}{3}(\overrightarrow{u} + \overrightarrow{v} + \overrightarrow{w}).$$
   Setting $t = \frac{3}{4}$ gives the centroid of the tetrahedron.

5. Given parallelogram $ABCD$ in the plane, show that $AC^2 + BD^2 = 2(AB^2 + AD^2)$. (Here, $\overline{AC}$ and $\overline{BD}$ are diagonals.) Hint: use vectors. You may assume that one of the vertices is at the origin.

Solution: Assume that A is located at the origin. Write $\overrightarrow{v} = B$ and $\overrightarrow{w} = D$. Then $C = \overrightarrow{v} + \overrightarrow{w}$, and the vector $\overline{AC}$ is $\overrightarrow{v} + \overrightarrow{w}$. Similarly, the vectors $\overline{BD} = \overrightarrow{w} - \overrightarrow{v}$, $\overline{AB} = \overrightarrow{v}$, and $\overline{AD} = \overrightarrow{w}$. We get

$$\begin{aligned}
AC^2 + BD^2 &= \|\overrightarrow{v} + \overrightarrow{w}\|^2 + \|\overrightarrow{w} - \overrightarrow{v}\|^2 \\
&= (\overrightarrow{v} + \overrightarrow{w}) \cdot (\overrightarrow{v} + \overrightarrow{w}) + (\overrightarrow{w} - \overrightarrow{v}) \cdot (\overrightarrow{w} - \overrightarrow{v}) \\
&= \|\overrightarrow{v}\|^2 + 2\overrightarrow{v} \cdot \overrightarrow{w} + \|\overrightarrow{w}\|^2 + \|\overrightarrow{w}\|^2 - 2\overrightarrow{v} \cdot \overrightarrow{w} + \|\overrightarrow{v}\|^2 \\
&= 2(\|\overrightarrow{v}\|^2 + \|\overrightarrow{w}\|^2) \\
&= 2(AB^2 + AD^2).
\end{aligned}$$