Name:

Math 410: Exam 1

Make sure to show all your work as clearly as possible. This includes justifying your answers if required. Compass and straightedge is allowed (and necessary!).

Please do not put any work on the back of pages. Use the space on the last page instead.

1. (15 pts) Show that the following axiomatic system is inconsistent. This system refers to the undefined terms *point* and *line*.

   I. Every line is a set of exactly 3 points.
   II. There are exactly 4 points.
   III. Each point is contained in a line.
   IV. No point is contained in more than one line.

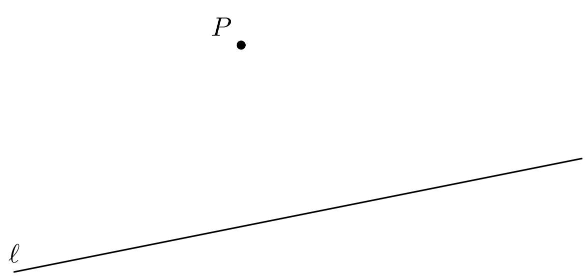
Solution: By II, there are exactly 4 points; call these P_1, P_2, P_3, P_4. By III, P_1 is contained in a line ℓ. By I, ℓ contains 3 points. Certainly P_1 is one of these points; without loss of generality the other two are P_2 and P_3.

Now consider P_4. By III, it is contained in a line ℓ'. If $\ell = \ell'$, then ℓ contains at least 4 points, which contradicts I. Therefore $\ell \neq \ell'$. By I, ℓ' contains 2 other points. Call one of these P_i, where $1 \leq i \leq 3$. But then $P_i \in \ell$ and $P_i \in \ell'$, contradicting IV. Thus the system is inconsistent.

2. (10 pts) Show that the following axiomatic system is consistent. This system refers to the undefined terms *fruit* and *bunch*.

   I. Every bunch is a set of fruit.
   II. Every pair of distinct bunches has exactly one fruit in common.
   III. Every fruit belongs to exactly two bunches.
   IV. There are exactly 4 bunches.

3. (10 pts) Below you are given a line ℓ and a point P. Construct a line through P which is perpendicular to ℓ. Additionally, describe the steps you used in the construction in order. You do not have to prove that your construction is correct.

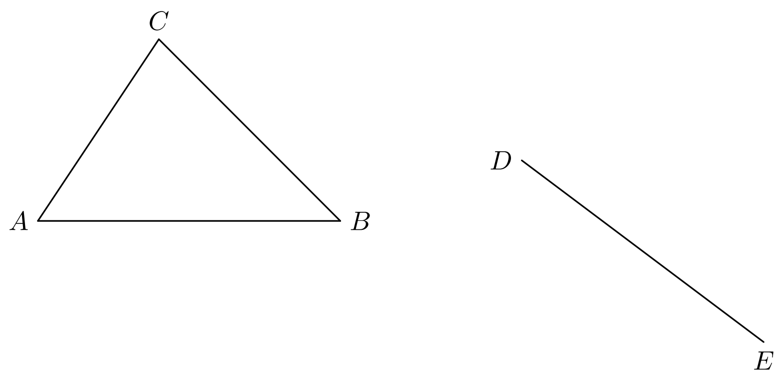


Solution: The construction is as follows:

1. Choose a point A on the line.
2. Construct a circle with center P and radius AP. Let B be the other intersection point of the circle and ℓ.

3. Construct a circle with center A and radius AP, and a circle with center B and radius AP.
4. Note that P is one intersection point of these two circles; let Q be the other. Construct $\overleftrightarrow{PQ}$, which is the perpendicular bisector.

4. (15 pts) Below you are given $\triangle ABC$ and a segment $\overline{DE}$ satisfying $AB = DE$. Construct F so that $\triangle ABC \cong \triangle DEF$. Additionally, describe the steps you used in the construction in order, and then prove that the construction is correct.



Solution:

1. Construct a circle with center D and radius AC.
2. Construct a cirlce with center E and radius BC.
3. Let F be either intersection point of the two circles.

Since F lies on the first circle, $DF = AC$. Since F lies on the second circle, $EF = BC$. We are given that $DE = AB$. By SSS, $\triangle ABC \cong \triangle DEF$.

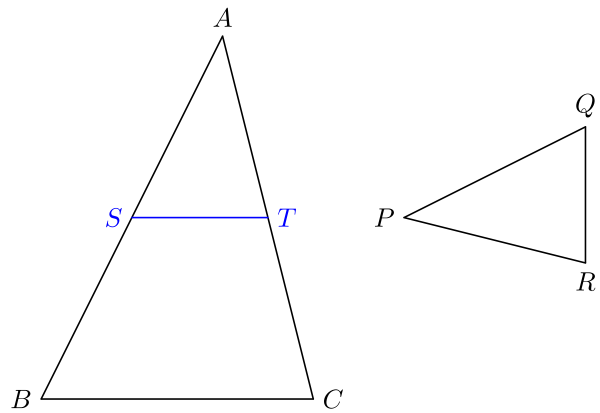
5. (15 pts) Suppose you are give a segment length 1. Explain how to construct a segment with length $\sqrt{1+\sqrt{2}}$. Describe the construction and prove that it is correct. You do not have to do the construction, but you ought to draw a picture demonstrating it.

Solution: There are a couple different ways to do this. The straightforward method is as follows:

1. Construct points P, Q, R on a line with Q between P and R and $PQ = 2$, $QR = 1$.
2. Construct the midpoint M of PR using the perpendicular bisector construction.
3. Construct the circle of radius MP and center M.
4. Construct a perpendicular to PR at Q; let one of the intersection points with the circle be S. Then as shown in the text, $QS = \sqrt{2}$.
5. Extend QS by 1 unit to get a segment of length $1 + \sqrt{2}$.
6. Repeat the procedure with QS in place of QR.

6. (15 pts) Suppose we are given $\triangle ABC$ and $\triangle PQR$ with $\angle A \cong \angle P$ and $\frac{AB}{PQ} = \frac{AC}{PR}$. Prove that the triangles are similar. (Hint: you may use the converse to Thales' Theorem.)

Solution: Without loss of generality $AB \geq PQ$, and hence $AC \geq PR$. Construct S between A and B with $AS = PQ$, and T between A and C with $AT = PR$.



Substituting, we have

$$\frac{AB}{AS} = \frac{AC}{AT}$$

and so by the converse to Thales', $\overline{ST} \parallel \overline{BC}$. Therefore we get the congruent corresponding angles $\angle AST \cong \angle ABC$ and $\angle ATS \cong \angle ACB$. Certainly $\angle BAC \cong \angle SAT$ (they are the same angle!) and so $\triangle ABC$ and $\triangle AST$ are similar.

But by SAS, $\triangle PQR \cong \triangle AST$, so the angles of these two triangles match up, and hence match up with the angles of $\triangle ABC$. It follows that $\triangle PQR$ and $\triangle ABC$ are similar.