Practice Final Selected Solutions

2C.3 A basis is $B = (x-6, x(x-6), x^2(x-6), x^3(x-6))$. For linear independence one can use the usual degree argument: suppose

$$a_1(x-6) + a_2 x(x-6) + a_3 x^2(x-6) + a_4 x^3(x-6) = 0.$$

Comparing the coefficient of x^4 on both sides, we get $a_4 = 0$. Then we repeat with x^3 and so on—details are left to the reader.

Next we show $\text{Span}(B) \subseteq U$. Plugging in 6 into the 4 polynomials in B clearly yields 0, so each element of B lies in U. Since Span(B) is the smallest subspace containing the entries of B, we have $\text{Span}(B) \subseteq U$.

As B is linearly independent, $\dim U \geq 4$. If $\dim U = 5$, then $U = \mathscr{P}_4(\mathbb{R})$. But $1 \in \mathscr{P}_4(\mathbb{R}) \setminus U$ (plugging in 6 into 1 yields 1), and hence $U \neq \mathscr{P}_4(\mathbb{R})$. Therefore $\dim U = 4$. It further follows that $\text{Span}(B) = U$, since they have the same dimension and one is a subspace of the other.

Lastly, we already showed $1 \in \mathscr{P}_4(\mathbb{R}) \setminus U$, and so from the contrapositive of Lemma 2.19, $(1, x-6, x(x-6), x^2(x-6), x^3(x-6))$ is linearly independent. Since this new list has 5 elements, it forms a basis for $\mathscr{P}_4(\mathbb{R})$. Additionally, the proof of Prop 2.33 shows that setting $W = \text{Span}(1)$, we do indeed get

$$\mathscr{P}_4(\mathbb{R}) = U \oplus W.$$

(I refer to the proof, since in that proof we extend a basis for U to a basis for the whole space, and then take W to be the span of the new vectors in the extended basis. That is exactly what we did above.)

3B.11 Let $Z = \text{null}(T)$. By Prop 2.33, $\exists$ a subspace U of V such that $V = U \oplus Z$. Let $w \in \text{Im}(T)$. By definition of image, $\exists v \in V$ such that $T(v) = w$. Since $V = U \oplus Z$, $\exists u \in U, z \in Z$ such that $v = u + z$. We have

$$\begin{aligned} T(u) &= T(v-z) \\ &= T(v) - T(z) \text{ by linearity of } T \\ &= w - 0 \text{ since } z \in \text{null}(T) \\ &= w. \end{aligned}$$

Thus $T(u) = w$, and recall we had $u \in U$. Since $w \in \text{Im}(T)$ was arbitrary, the claim is proven.

5A.2 Let $v \in V_1 + \cdots + V_m$. By definition of sum of subspaces, for $i = 1, \ldots, m$, $\exists v_i \in V_i$ such that

$$v = v_1 + v_2 + \cdots + v_m.$$

By linearity of T,

$$T(v) = T(v_1) + T(v_2) + \cdots + T(v_m).$$

Let $u_i = T(v_i)$ for each i. Since $v_i \in V_i$ and V_i is invariant under T, we must have $u_i \in V_i$ as well. Thus

$$T(v) = u_1 + u_2 + \cdots + u_m \in V_1 + V_2 + \cdots + V_m.$$

Therefore $T(v) = V_1 + \cdots + V_m$, which shows invariance.

5A.12 The eigenvalues are 0 and 1. For 0, every nonzero element of W is an eigenvector. For 1, every nonzero element of U is an eigenvector.

To prove this, suppose λ is an eigenvalue with eigenvector v. Thus

$$T(v) = \lambda v.$$

Since $V = U \oplus W$, $\exists$ unique $u \in U, w \in W$ such that $v = u + w$. We know $T(u + w) = u$, so plugging in to the above equation, we get

$$u = \lambda u + \lambda w,$$

or

$$(1 - \lambda)u = \lambda w.$$

The left side is in U, and the right side is in W. Since $U \cap W = \{0\}$ by Prop 1.46, we must have either $\lambda = 0$ and $u = 0$, or $\lambda = 1$ and $w = 0$. In the first case, w can be anything nonzero in W, and so any nonzero vector in W is an eigenvector for 0. If $\lambda = 1$, then u can be any nonzero vector in U, and will be an eigenvector for 1.