Section 2C Selected Solutions

2C.8 Let $U = \text{Span}(v_1 + w, v_2 + w, \ldots, v_m + w)$. Let $u_1, u_2, \ldots, u_n$ be a basis for $\text{Span}(v_1 + w, v_2 + w, \ldots, v_m + w)$, so $n = \dim U$. Let $W = U + \text{Span}(w) = \text{Span}(u_1, \ldots, u_n, w)$. We have

$$\begin{aligned} W &= \text{Span}(v_1 + w, v_2 + w, \ldots, v_m + w, w) \\ &= \text{Span}(v_1, \ldots, v_m, w). \end{aligned}$$

The last equality follows since, on the one hand, clearly

$$v_j + w \in \text{Span}(v_1, \ldots, v_m, w)\ \forall j,$$

and so $\text{Span}(v_1 + w, \ldots, v_m + w, w) \subseteq \text{Span}(v_1, \ldots, v_m, w)$; while on the other hand,

$$v_j = (v_j + w) - w\ \forall j,$$

giving the reverse containment.

Now $\text{Span}(v_1, \ldots, v_m) \subseteq \text{Span}(v_1, \ldots, v_m, w) = W$, so $m \leq \dim W$. If $w \in \text{Span}(u_1, \ldots, u_n)$, then $W = U$ (details left to reader) and $\dim W = n$. In this case, $n \geq m > m - 1$, and we are done. If $w \notin \text{Span}(u_1, \ldots, u_n)$, then by the Linear Dependence Lemma, $u_1, \ldots, u_n, w$ is linearly dependent; the list also spans W by definition of W, and hence $\dim W = n + 1$. Thus $m \leq n + 1$, and so $n \geq m - 1$ in this case as well.

2C.9 We apply induction on m to prove the list of polynomials is linearly independent. The $m = 0$ case is obvious, since the basis element is a single nonzero constant. For the inductive case, suppose the claim is true for $\mathscr{P}_{m-1}(\mathbb{F})$. Let the lead coefficient of p_k be c_k; by definition of lead coefficient, $c_k \neq 0$. Now suppose

$$a_0 p_0 + a_1 p_1 + \cdots + a_m p_m = 0$$

for $a_i \in \mathbb{F}$. The coefficient of x^m on the left side is $a_m c_m$ by our hypothesis. But on the right, it is zero. Therefore $a_m c_m = 0$. As $c_m \neq 0$, we have $a_= 0$. Thus

$$a_0 p_0 + \cdots + a_{m-1} p_{m-1} = 0.$$

But now we can apply our inductive hypothesis to conclude that $a_j = 0$ for all j. Therefore $p_0, \ldots, p_m$ is linearly independent.

Since the number of polynomials equals the dimension of $\mathscr{P}_m(\mathbb{F})$, $p_0, \ldots, p_m$ is a basis.

2C.18 Let $v_1, \ldots, v_n$ be a basis for V, and set $V_j = \text{Span}(v_j)$ for all j. We first show that $V = V_1 + V_2 + \cdots + V_n$. For let $v \in V$. Since $v_1, \ldots, v_n$ is a basis for V, $\exists a_1, \ldots, a_n \in \mathbb{F}$ such that

$$v = a_1 v_1 + \cdots + a_n v_n.$$

Let $w_j = a_j v_j$ for each j. Observe that $w_j \in V_j$, and that $v = w_1 + \cdots + w_n$. Thus $V = V_1 + \cdots + V_n$ as required.

Next we need to show the uniqueness. Let w_j be as above, and also suppose $u_1, \ldots, u_n$ satisfy $u_j \in V_j$ $\forall j$ and $v = u_1 + \cdots + u_n$. By definition of V_j, $\exists b_j \in \mathbb{F}$ such that $u_j = b_j v_j$ for each j, and hence

$$v = b_1 v_1 + \cdots + b_n v_n.$$

By Prop 2.28, the coefficients b_j are unique, and in particular $b_j = a_j$ for all j. It follows that $b_j v_j = a_j v_j$, and hence $w_j = u_j$ for all j. The uniqueness of the w_j follows, and therefore the sum is a direct sum.