Homework 9 Selected Solutions

Due: Tuesday, April 7

3B.13 By Rank-Nullity,

$$\dim \mathbb{R}^8 = \dim U + \dim \operatorname{Im}(T).$$

Thus $\dim \operatorname{Im}(T) = 8 - 3 = 5$. As $\operatorname{Im}(T)$ is a subspace of $\mathbb{R}^5$ with the same dimension, we must have $\operatorname{Im}(T) = \mathbb{R}^5$. Therefore T is surjective.

3B.19 First suppose $\exists S \in \mathcal{L}(W, V)$ such that ST is the identity. Suppose $x, y \in V$ satisfy $T(x) = T(y)$. Applying S to both sides yields $S(T(x)) = S(T(y))$, so $(ST)(x) = (ST)(y)$. But ST is the identity, so we obtain $x = y$. Therefore T is injective.

Now suppose T is injective. **Note: this proof has been updated. The previous version used Rank-Nullity on T. But we don't know that V is finite-dimensional, and hence cannot use Rank-Nullity.** We know $\operatorname{Im}(T)$ is a subspace of W. Since W is finite-dimensional, so is $\operatorname{Im}(T)$. Let $w_1, \dots, w_m$ be a basis for $\operatorname{Im}(T)$. Extend this to a basis $w_1, \dots, w_m, u_1, \dots, u_n$ for W. By definition of image, $\forall j$, $\exists v_j \in V$ such that $T(v_j) = w_j$. Now we use Prop 3.4 to define linear $S : W \to V$ by

$$\begin{aligned} S(w_j) &= v_j \text{ for } j = 1, \dots, m \\ S(u_k) &= 0 \text{ for } k = 1, \dots, n. \end{aligned}$$

Now we show that $ST = \mathrm{id}_V$. Let $v \in V$, and let $w = T(v)$. Then $w \in \operatorname{Im}(T)$, and since $(w_1, \dots, w_m)$ forms a basis for $\operatorname{Im}(T)$, $\exists a_1, \dots, a_m \in \mathbb{F}$ such that

$$w = a_1 w_1 + \cdots + a_m w_m.$$

Observe that

$$\begin{aligned} T(a_1 v_1 + \cdots + a_m v_m) &= a_1 T(v_1) + \cdots + a_m T(v_m) \\ &= a_1 w_1 + \cdots + a_m w_m \\ &= w, \end{aligned}$$

and so

$$T(a_1 v_1 + \cdots + a_m v_m) = T(v).$$

Since T is injective, we must have

$$v = a_1 v_1 + \cdots + a_m v_m.$$

Next, by definition of S and linearity, we have

$$\begin{aligned} (ST)(v) &= S(w) \\ &= S(a_1 w_1 + \cdots + a_m w_m) \\ &= a_1 v_1 + \cdots + a_m v_m \\ &= v. \end{aligned}$$

Since v was arbitrary, we conclude that $ST = \mathrm{id}_V$.

3C.5 We first pick a basis for V. Let $u_1, \dots, u_m$ be a basis for null(T). Extend this to a basis $B = (v_1, \dots, v_n, u_1, \dots, u_m)$ for V. Note that by Rank-Nullity, $n = \dim(\mathrm{Im}(T))$.

Now we pick a basis for W. Let $w_j = T(v_j)$ for $j = 1, \dots, n$. I claim that $w_1, \dots, w_n$ is linearly independent. (In fact, the list forms a basis for Im(T), but we will not prove it.) For if $a_1, \dots, a_n \in \mathbb{F}$ satisfy

$$a_1 w_1 + \cdots + a_n w_n = 0,$$

Then

$$\begin{aligned} T(a_1 v_1 + \cdots + a_n v_n) &= a_1 T(v_1) + \cdots + a_n T(v_n) \\ &= a_1 w_1 + \cdots + a_n w_n \\ &= 0, \end{aligned}$$

and hence $a_1 v_1 + \cdots + a_n v_n \in$ null(T). Thus $\exists b_1, \dots, b_m \in \mathbb{F}$ such that

$$a_1 v_1 + \cdots + a_n v_n = b_1 u_1 + \cdots + b_m u_m,$$

or

$$a_1 v_1 + \cdots + a_n v_n - b_1 u_1 - \cdots - b_m u_m = 0.$$

Since B is linearly independent, the $a_j = 0$ for all j. This shows that $(w_1, \dots, w_n)$ is linearly independent. We complete to a basis for W,

$$B' = (w_1, \dots, w_n, t_1, \dots, t_\ell).$$

Now we construct the matrix for T with respect to the bases B, B'. As $T(v_j) = w_j$, we have that the jth column of $\mathcal{M}(T)$ is the vector with 1 in the jth row and zeros elsewhere. As $T(u_k) = 0$, all columns after the nth are entirely zero. Finally, recall that $n = \dim(\mathrm{Im}(T))$. The claim follows.

Supplemental solutions. These were not graded.

3C.2 First we construct our basis for V. Let $\dim V = n$. From Rank-Nullity and $\dim \mathrm{Im}(T) = 1$, we seen that $\dim \mathrm{null}(T) = n - 1$. Let $u_1, \dots, u_{n-1}$ be a basis for null(T). Extend to a basis $u_1, \dots, u_{n-1}, v_n$ for V. Finally, set $v_j = u_j + v_n$ for $1 \leq j \leq n - 1$.

Observe that $u_j = v_j - v_n$ for $1 \leq j \leq n - 1$. It follows that

$$\mathrm{Span}(u_1, \dots, u_{n-1}, v_n) \subseteq \mathrm{Span}(v_1, \dots, v_n).$$

But $\mathrm{Span}(u_1, \dots, u_{n-1}, v_n) = V$, so $\mathrm{Span}(v_1, \dots, v_n) = V$. Since $n = \dim V$, $v_1, \dots, v_n$ is a basis for V.

Additionally, observe that for $1 \leq j \leq n - 1$,

$$T(v_j) = T(u_j + v_n) = T(u_j) + T(v_n) = T(v_n)$$

since $u_j \in \text{null}(T)$. We will use this fact later.

Next we choose our basis for W. Let $w = T(v_n)$. Since $v_n \notin \text{null}(T)$, $w \neq 0$. Extend w to a basis $w, w_2, \ldots, w_m$ for W; thus $m = \dim W$. Set

$$w_1 = w - w_2 - \cdots - w_m.$$

I claim that $w_1, \ldots, w_m$ is a basis for W. Certainly $w_k \in \text{Span}(w_1, \ldots, w_m)$ for $2 \leq k \leq m$. But

$$w = w_1 + w_2 + \cdots + w_m,$$

so $w \in \text{Span}(w_1, \ldots, w_m)$. Therefore

$$\text{Span}(w, w_2, \ldots, w_m) \subseteq \text{Span}(w_1, w_2, \ldots, w_m).$$

Since the first span is W, the second span must equal W. As $m = \dim W$, $w_1, \ldots, w_m$ forms a basis for W.

Finally, we construct the matrix $\mathcal{M}(T)$ for T with respect to these two bases. We have

$$T(v_1) = T(v_n) = w = w_1 + \cdots + w_m,$$

and so the first column of $\mathcal{M}(T)$ is all 1s. But $T(v_k) = T(v_n)$ for all k, $1 \leq k \leq n$, and hence the same is true for the remaining columns. Therefore every entry of $\mathcal{M}(T)$ is 1.

3C.4 Let $x^3, x^2, x, 1$ be our basis for $\mathscr{P}_3(\mathbb{R})$; this is just the standard basis in reverse order. Let $3x^2, 2x, 1$ be our basis for $\mathscr{P}_2(\mathbb{R})$; it's just the standard basis in reverse order, and each basis element is multiplied by a nonzero scalar. We have

$$\begin{aligned} D(x^3) &= 1 \cdot 3x^2 + 0 \cdot 2x + 0 \cdot 1 \\ D(x^2) &= 0 \cdot 3x^2 + 1 \cdot 2x + 0 \cdot 1 \\ D(x) &= 0 \cdot 3x^2 + 0 \cdot 2x + 1 \cdot 1 \\ D(1) &= 0 \cdot 3x^2 + 0 \cdot 2x + 0 \cdot 1. \end{aligned}$$

The matrix for D with respect to our bases is therefore the one given in the problem.