Homework 8 Selected Solutions
Due: Tuesday, March 24

3A.12 We will show the additivity condition does not hold. But first we must come up with the two vectors we wish to add! The hypothesis that $S \neq 0$ means that $\exists u \in U$ such that $S(u) \neq 0$. This u will be our first vector. Since $U \neq V$, $\exists w \in V \setminus U$. As $0 \in U$ since U is a subspace, we know that $w \neq 0$. This w is our second vector. I first claim that $u + w \notin U$. To show this, we assume for the sake of contradiction that $u + w \in U$. Then since U is closed under scalar multiplication and vector addition,

$$w = (u + w) + (-1)u \in U.$$

This is a contradiction of our choice of w. Therefore $u + w \notin U$.

From our definition of T, we have

$$T(u) = S(u) \neq 0,$$
$$T(w) = 0, \text{ and}$$
$$T(u + w) = 0.$$

But also
$$T(u) + T(w) = S(u) + 0 = S(u) \neq 0.$$

Therefore $T(u + w) \neq T(u) + T(w)$, and hence T is not linear.

3A.15 We need to pick our w_i. To do this, we have to use our linear dependence relation on the v_i. It's actually easiest to use the linear dependence lemma. That lemma implies that $\exists k$, $1 \leq k \leq m$, and scalars $a_1, \ldots, a_{k-1}$ such that
$$v_k = a_1 v_1 + \cdots + a_{k-1} v_{k-1}. \tag{1}$$

Now we choose our w_i. Let $w \in W$ be any nonzero vector; we know there is one since $W \neq \{0\}$. Let $w_k = w$, and for $i \neq k$, let $w_i = 0$.

I claim there is no linear map $T : V \to W$ satisfying $T(v_i) = w_i$ for all i. For the sake of contradiction, suppose there were such a T. Then

$$
\begin{aligned}
w &= w_k \\
&= T(v_k) \\
&= T(a_1 v_1 + \cdots + a_{k-1} v_{k-1}) \\
&= a_1 T(v_1) + \cdots + a_{k-1} T(v_{k-1}) \text{ by linearity of } T \\
&= a_1 \cdot 0 + \cdots + a_{k-1} \cdot 0 \\
&= 0.
\end{aligned}
$$

By our choice of w, $w \neq 0$. Thus we have a contradiction, and hence there is no such T.

By the way, notice that the proof works even if $k = 1$. For then equation (1) becomes $v_1 = 0$, and we'd have $w = w_1 = T(v_1) = T(0) = 0$, so the proof as written still works.

3B.9 Suppose $a_1, \ldots, a_n$ are scalars satisfying

$$a_1 T v_1 + \cdots + a_n T v_n = 0.$$

By linearity of T,
$$T(a_1 v_1 + \cdots + a_n v_n) = 0.$$

In particular, $a_1 v_1 + \cdots + a_n v_n \in \text{null}(T)$. But T is injective, so $\text{null}(T) = \{0\}$. Hence
$$a_1 v_1 + \cdots + a_n v_n = 0.$$

By hypothesis, $v_1, \ldots, v_n$ is linearly independent, so $a_i = 0$ for all i. Thus $T v_1, \ldots, T v_n$ is linearly independent.